Lecture  11

Semigroups of Operators

In this Lecture we gather a few notions on one-parameter semigroups of linear operators, confining to the essential tools that are needed in the sequel. As usual, \( X \) is a real or complex Banach space, with norm \( \| \cdot \| \). In this lecture Gaussian measures play no role.

11.1 Strongly continuous semigroups

Definition 11.1.1. Let \( \{ T(t) : t \geq 0 \} \) be a family of operators in \( \mathcal{L}(X) \). We say that it is a semigroup if
\[
T(0) = I, \quad T(t+s) = T(t)T(s) \quad \forall \ t, s \geq 0.
\]
A semigroup is called strongly continuous (or \( C_0 \)-semigroup) if for every \( x \in X \) the function \( T(\cdot)x : [0, \infty) \to X \) is continuous.

Let us present the most elementary properties of strongly continuous semigroups.

Lemma 11.1.2. Let \( \{ T(t) : t \geq 0 \} \subset \mathcal{L}(X) \) be a semigroup. The following properties hold:

(a) if there exist \( \delta > 0, M \geq 1 \) such that
\[
\| T(t) \| \leq M, \quad 0 \leq t \leq \delta,
\]
then, setting \( \omega = (\log M)/\delta \) we have
\[
\| T(t) \| \leq Me^{\omega t}, \quad t \geq 0. \tag{11.1.1}
\]
Moreover, for every \( x \in X \) the function \( t \mapsto T(t)x \) is continuous in \( [0, +\infty) \) iff it is continuous at 0.

(b) If \( \{ T(t) : t \geq 0 \} \) is strongly continuous, then for any \( \delta > 0 \) there is \( M_\delta > 0 \) such that
\[
\| T(t) \| \leq M_\delta, \quad \forall \ t \in [0, \delta].
\]
Proof. (a) Using repeatedly the semigroup property in Definition 11.1.1 we get $T(t) = T(\delta)^{n-1}T(t-\delta)$ for $(n-1)\delta \leq t \leq n\delta$, whence $\|T(t)\| \leq M^n \leq M e^{\omega t}$. Let $x \in X$ be such that $t \mapsto T(t)x$ is continuous at 0, i.e., $\lim_{h \to 0^+} T(h)x = x$. Using again the semigroup property in Definition 11.1.1 it is easily seen that for every $t > 0$ the equality $\lim_{h \to 0^+} T(t+h)x = T(t)x$ holds. Moreover,

$$\|T(t-h)x - T(t)x\| = \|T(t-h)(x - T(h)x)\| \leq M e^{\omega(t-h)}\|(x - T(h)x)\|, \quad 0 < h < t,$$

whence $\lim_{h \to 0^+} T(t-h)x = T(t)x$. It follows that $t \mapsto T(t)x$ is continuous in $[0, +\infty)$.

(b) Let $x \in X$. As $T(t)x$ is continuous, for every $\delta > 0$ there is $S_{\delta,\varepsilon} > 0$ such that

$$\|T(t)x\| \leq S_{\delta,\varepsilon}, \quad \forall \ t \in [0, \delta].$$

The statement follows from the Uniform Boundedness Principle, see e.g. [Br, Chapter 2] or [DS1, §II.1].

If (11.1.1) holds with $M = 1$ and $\omega = 0$ then the semigroups is said semigroup of contractions or contractive semigroup. From now on, $\{T(t) : t \geq 0\}$ is a fixed strongly continuous semigroup.

Definition 11.1.3. The infinitesimal generator (or, shortly, the generator) of the semigroup $\{T(t) : t \geq 0\}$ is the operator defined by

$$D(L) = \left\{ x \in X : \exists \lim_{h \to 0^+} \frac{T(h) - I}{h}x \right\}, \quad Lx = \lim_{h \to 0^+} \frac{T(h) - I}{h}x.$$

By definition, the vector $Lx$ is the right derivative of the function $t \mapsto T(t)x$ at $t = 0$ and $D(L)$ is the subspace where such derivative exists. In general, $D(L)$ is not the whole $X$, but it is dense, as the next proposition shows.

Proposition 11.1.4. The domain $D(L)$ of the generator is dense in $X$.

Proof. Set

$$M_{a,t}x = \frac{1}{t} \int_a^{a+t} T(s)x \, ds, \quad a \geq 0, \ t > 0, \ x \in X$$

(this is a $X$-valued Bochner integral). As the function $s \mapsto T(s)x$ is continuous, we have (see Exercise 11.1)

$$\lim_{t \to 0} M_{a,t}x = T(a)x.$$

In particular, $\lim_{t \to 0^+} M_{0,t}x = x$ for every $x \in X$. Let us show that for every $t > 0$, $M_{0,t}x \in D(L)$, which implies that the statement holds. We have

$$\frac{T(h) - I}{h} M_{0,t}x = \frac{1}{ht} \left( \int_0^t T(h+s)x \, ds - \int_0^t T(s)x \, ds \right)$$

$$= \frac{1}{ht} \left( \int_h^{h+t} T(s)x \, ds - \int_0^t T(s)x \, ds \right)$$

$$= \frac{1}{ht} \left( \int_{t-h}^h T(s)x \, ds - \int_0^h T(s)x \, ds \right)$$

$$= \frac{M_{t,h}x - M_{0,h}x}{t}.$$
Therefore, for every \( x \in X \) we have \( M_{0,t}x \in D(L) \) and

\[
L M_{0,t}x = \frac{T(t)x - x}{t}.
\]

Proposition 11.1.5. For every \( t > 0 \), \( T(t) \) maps \( D(L) \) into itself, and \( L \) and \( T(t) \) commute on \( D(L) \).

If \( x \in D(L) \), then the function \( T(\cdot)x \) is differentiable at every \( t \geq 0 \) and

\[
\frac{d}{dt} T(t)x = LT(t)x = T(t)Lx, \quad t \geq 0.
\]

Proof. For every \( x \in X \) and for every \( h > 0 \) we have

\[
\frac{T(h) - I}{h} T(t)x = T(t) \frac{T(h) - I}{h} x.
\]

If \( x \in D(L) \), letting \( h \to 0 \) we obtain \( T(t)x \in D(L) \) and \( LT(t)x = T(t)Lx \).

Fix \( t_0 \geq 0 \) and let \( h > 0 \). We have

\[
\frac{T(t_0 + h)x - T(t_0)x}{h} = T(t_0) \frac{T(h) - I}{h} x \to T(t_0)Lx \quad \text{as} \quad h \to 0^+.
\]

This shows that \( T(\cdot)x \) is right differentiable at \( t_0 \). Let us show that it is left differentiable, assuming \( t_0 > 0 \). If \( h \in (0, t_0) \) we have

\[
\frac{T(t_0 - h)x - T(t_0)x}{-h} = T(t_0 - h) \frac{T(h) - I}{h} x \to T(t_0)Lx \quad \text{as} \quad h \to 0^+,
\]
as

\[
\left\| T(t_0 - h) \frac{T(h) - I}{h} x - T(t_0)Lx \right\| \leq \left\| T(t_0 - h) \left( \frac{T(h) - I}{h} x - Lx \right) \right\| + \left\| (T(t_0 - h) - T(t_0))Lx \right\|
\]

and \( \|T(t_0 - h)\| \leq \sup_{0 \leq t \leq t_0} \|T(t)\| < \infty \) by Lemma 11.1.2. It follows that the function \( t \mapsto T(t)x \) is differentiable at all \( t \geq 0 \) and its derivative is \( T(t)Lx \), which is equal to \( LT(t)x \) by the first part of the proof.

Using Proposition 11.1.5 we prove that the generator \( L \) is a closed operator. Therefore, \( D(L) \) is a Banach space with the graph norm \( \|x\|_{D(L)} = \|x\| + \|Lx\| \).

Proposition 11.1.6. The generator \( L \) of any strongly continuous semigroup is a closed operator.

Proof. Let \( (x_n) \) be a sequence in \( D(L) \), and let \( x, y \in X \) be such that \( x_n \to x \), \( Lx_n =: y_n \to y \). By Proposition 11.1.5 the function \( t \mapsto T(t)x_n \) is continuously differentiable in \([0, \infty)\). Hence for \( 0 < h < 1 \) we have (see Exercise 11.1)

\[
\frac{T(h) - I}{h} x_n = \frac{1}{h} \int_0^h LT(t)x_n dt = \frac{1}{h} \int_0^h T(t)y_n dt,
\]
and then
\[
\left\| \frac{T(h) - I}{h}x - y \right\| \leq \left\| \frac{T(h) - I}{h} (x - x_n) \right\| + \left\| \frac{1}{h} \int_0^h T(t)(y_n - y) dt \right\| + \left\| \frac{1}{h} \int_0^h T(t)y dt - y \right\|
\]
\[
\leq C + \frac{1}{h} \left\| x - x_n \right\| + C \left\| y_n - y \right\| + \frac{1}{h} \int_0^h T(t)y dt - y,
\]
where \( C = \sup_{0 < t < 1} \| T(t) \|. \) Given \( \varepsilon > 0, \) there is \( h_0 \) such that for \( 0 < h \leq h_0 \) we have \( \| \int_0^h T(t)y dt/h - y \| \leq \varepsilon/3. \) For \( h \in (0, h_0], \) take \( n \) such that \( \| x - x_n \| \leq \varepsilon h/3(C + 1) \) and \( \| y_n - y \| \leq \varepsilon/3C: \) we get \( \| T(h) - I/h x - y \| \leq \varepsilon \) and therefore \( x \in D(L) \) and \( y = Lx, \) i.e., the operator \( L \) is closed.

Proposition 11.1.5 implies that for any \( x \in D(L) \) the function \( u(t) = T(t)x \) is differentiable for \( t \geq 0 \) and it solves the Cauchy problem

\[
\begin{cases}
  u'(t) = Lu(t), & t \geq 0, \\
  u(0) = x.
\end{cases}
\]

Lemma 11.1.7. For every \( x \in D(L), \) the function \( u(t) := T(t)x \) is the unique solution of (11.1.3) belonging to \( C([0, +\infty); D(L)) \cap C^1([0, +\infty); X). \)

Proof. From Proposition 11.1.5 we know that \( u'(t) = T(t)Lx \) for every \( t \geq 0, \) and then \( u' \in C([0, +\infty); X). \) Therefore, \( u \in C^1([0, +\infty); X). \) Since \( D(L) \) is endowed with the graph norm, a function \( u : [0, +\infty) \to D(L) \) is continuous iff both \( u \) and \( Lu \) are continuous. In our case, both \( u \) and \( Lu = u' \) belong to \( C([0, +\infty); X), \) and then \( u \in C([0, +\infty); D(L)). \)

Let us prove that (11.1.3) has a unique solution in \( C([0, +\infty); D(L)) \cap C^1([0, +\infty); X). \) If \( u \in C([0, +\infty); D(L)) \cap C^1([0, +\infty); X) \) is any solution, we fix \( t > 0 \) and define the function

\[
v(s) := T(t-s)u(s), \quad 0 \leq s \leq t.
\]

Then (Exercise 11.2) \( v \) is differentiable, and \( v'(s) = -T(t-s)Lu(s) + T(t-s)u'(s) = 0 \) for \( 0 \leq s \leq t, \) whence \( v(t) = v(0), \) i.e., \( u(t) = T(t)x. \)

Remark 11.1.8. If \( \{T(t) : t \geq 0\} \) is a \( C_0 \)-semigroup with generator \( L, \) then for every \( \lambda \in \mathbb{C} \) the family of operators

\[
S(t) = e^{\lambda t}T(t), \quad t \geq 0,
\]
is a \( C_0 \)-semigroup as well, with generator \( L + \lambda I : D(L) \to X. \) The semigroup property is obvious. Concerning the generator, for every \( x \in X \) we have

\[
\frac{S(h)x - x}{h} = e^{\lambda h}T(h)x - x + \frac{e^{\lambda h}x - x}{h}
\]
and then

\[
\lim_{h \to 0^+} \frac{S(h)x - x}{h} = \lim_{h \to 0^+} e^{\lambda h}T(h)x - x + \frac{e^{\lambda h}x - x}{h} = Lx + \lambda x
\]
iff \( x \in D(L). \)
Semigroups of Operators

Let \( \{T(t) : t \geq 0\} \) be a strongly continuous semigroup. Characterising the domain of its generator \( L \) may be difficult. However, for many proofs it is enough to know that “good” elements \( x \) are dense in \( D(L) \). A subspace \( D \subset D(L) \) is called a core of \( L \) if \( D \) is dense in \( D(L) \) with respect to the graph norm. The following proposition gives an easily checkable sufficient condition in order that \( D \) is a core.

**Lemma 11.1.9.** If \( D \subset D(L) \) is a dense subspace of \( X \) and \( T(t)(D) \subset D \) for every \( t \geq 0 \), then \( D \) is a core.

**Proof.** Let \( M, \omega \) be such that \( \|T(t)\| \leq Me^{\omega t} \) for every \( t > 0 \). For \( x \in D(L) \) we have

\[
Lx = \lim_{t \to 0} \frac{1}{t} \int_0^t T(s)Lx \, ds.
\]

Let \( (x_n) \subset D \) be a sequence such that \( \lim_{n \to \infty} x_n = x \). Set

\[
y_{n,t} = \frac{1}{t} \int_0^t T(s)x_n \, ds = \frac{1}{t} \int_0^t T(s)(x_n - x) \, ds + \frac{1}{t} \int_0^t T(s)x \, ds.
\]

As the \( D(L) \)-valued function \( s \mapsto T(s)x_n \) is continuous in \([0, +\infty)\), the vector \( \int_0^t T(s)x_n \, ds \) belongs to \( D(L) \). Moreover, it is the limit of the Riemann sums of elements of \( D \) (see Exercise 11.1), hence it belongs to the closure of \( D \) in \( D(L) \). Therefore, \( y_{n,t} \) belongs to the closure of \( D \) in \( D(L) \) for every \( n \) and \( t \). Furthermore,

\[
\|y_{n,t} - x\| \leq \left\| \frac{1}{t} \int_0^t T(s)(x_n - x) \, ds \right\| + \left\| \frac{1}{t} \int_0^t T(s)x \, ds - x \right\|
\]

tends to 0 as \( t \to 0, n \to \infty \). By (11.1.2) we have

\[
Ly_{n,t} - Lx = \frac{T(t)(x_n - x) - (x_n - x)}{t} + \frac{1}{t} \int_0^t T(s)Lx \, ds - Lx.
\]

Given \( \varepsilon > 0 \), fix \( \tau > 0 \) such that

\[
\left\| \frac{1}{\tau} \int_0^\tau T(s)Lx \, ds - Lx \right\| \leq \varepsilon,
\]

and then take \( n \in \mathbb{N} \) such that \( (Me^{\omega \tau} + 1)\|x_n - x\|/\tau \leq \varepsilon \). Therefore, \( \|L_{y_n,t} - Lx\| \leq 2\varepsilon \) and the statement follows.

### 11.2 Generation Theorems

In this section we recall the main generation theorems for \( C_0 \)-semigroups. The most general result is the classical Hille–Yosida Theorem, which gives a complete characterisation of the generators. For contractive semigroups, i.e., semigroups verifying the estimate \( \|T(t)\| \leq 1 \) for all \( t \geq 0 \), the characterisation of the generators provided by the Lumer-Phillips Theorem is often useful. We do not present here the proofs of these results, referring e.g. to [EN, §II.3].
First, we recall the definition of \textit{spectrum} and \textit{resolvent}. The natural setting for spectral theory is that of complex Banach spaces, hence if $X$ is real we replace it by its complexification $\tilde{X} = \{x + iy : x, y \in X\}$ endowed with the norm
\[
\|x + iy\|_{\tilde{X}} := \sup_{-\pi \leq \theta \leq \pi} \|x \cos \theta + y \sin \theta\|
\]
(notice that the seemingly more natural “Euclidean norm” $(\|x\|^2 + \|y\|^2)^{1/2}$ is not a norm in general).

\textbf{Definition 11.2.1.} Let $L : D(L) \subset X \to X$ be a linear operator. The \textit{resolvent set} $\rho(L)$ and the \textit{spectrum} $\sigma(L)$ of $L$ are defined by
\[
\rho(L) = \{\lambda \in \mathbb{C} : \exists (\lambda I - L)^{-1} \in \mathcal{L}(X)\}, \quad \sigma(L) = \mathbb{C} \setminus \rho(L).
\]

The complex numbers $\lambda \in \sigma(L)$ such that $\lambda I - L$ is not injective are the eigenvalues, and the vectors $x \in D(L)$ such that $Lx = \lambda x$ are the eigenvectors (or eigenfunctions, when $X$ is a function space). The set $\sigma_p(L)$ whose elements are all the eigenvalues of $L$ is the \textit{point spectrum}.

For $\lambda \in \rho(L)$, we set
\[
R(\lambda, L) := (\lambda I - L)^{-1}.
\]

The operator $R(\lambda, L)$ is the \textit{resolvent operator} or briefly \textit{resolvent}.

We ask to check (Exercise 11.3) that if the resolvent set $\rho(L)$ is not empty, then $L$ is a closed operator. We also ask to check (Exercise 11.4) the following equality, known as the \textit{resolvent identity}
\[
R(\lambda, L) - R(\mu, L) = (\mu - \lambda)R(\lambda, L)R(\mu, L), \quad \forall \lambda, \mu \in \rho(L).
\]

\textbf{Theorem 11.2.2} (Hille–Yosida). The linear operator $L : D(L) \subset X \to X$ is the generator of a $C_0$-semigroup verifying estimate (11.1.1) iff the following conditions hold:
\[
\begin{cases}
(i) & D(L) \text{ is dense in } X, \\
(ii) & \rho(L) \supset \{\lambda \in \mathbb{R} : \lambda > \omega\}, \\
(iii) & \|R(\lambda, L)\|^k_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^k} \quad \forall k \in \mathbb{N}, \forall \lambda > \omega.
\end{cases}
\]

Before stating the Lumer–Phillips Theorem, we define the \textit{dissipative} operators.

\textbf{Definition 11.2.3.} A linear operator $(L, D(L))$ is called dissipative if
\[
\|(\lambda I - L)x\| \geq \lambda\|x\|
\]
for all $\lambda > 0$, $x \in D(L)$.

\textbf{Theorem 11.2.4} (Lumer–Phillips). A densely defined and dissipative operator $L$ on $X$ is closable and its closure is dissipative. Moreover, the following statements are equivalent.

(i) The closure of $L$ generates a contraction $C_0$-semigroup.

(ii) The range of $\lambda I - L$ is dense in $X$ for some (hence all) $\lambda > 0$.  

11.3 Invariant measures

In our lectures we shall encounter semigroups defined in $L^p$ spaces, i.e., $X = L^p(\Omega)$ where $(\Omega, \mathcal{F}, \mu)$ is a measure space, with $\mu(\Omega) < \infty$. A property that will play an important role is the conservation of the mean value, namely

$$\int_{\Omega} T(t)f \, d\mu = \int_{\Omega} f \, d\mu \quad \forall t > 0, \forall f \in L^p(\Omega).$$

In this case $\mu$ is called invariant for $T(t)$. The following proposition gives an equivalent condition for invariance, in terms of the generator of the semigroup rather than the semigroup itself.

**Proposition 11.3.1.** Let \( \{T(t) : t \geq 0\} \) be a strongly continuous semigroup with generator $L$ in $L^p(\Omega, \mu)$, where $(\Omega, \mu)$ is a measure space, $p \in [1, +\infty)$, and $\mu(\Omega) < \infty$. Then

$$\int_{\Omega} T(t)f \, d\mu = \int_{\Omega} f \, d\mu \quad \forall t > 0, \forall f \in L^p(\Omega, \mu) \iff \int_{\Omega} Lf \, d\mu = 0 \quad \forall f \in D(L).$$

**Proof.** “⇒” Let $f \in D(L)$. Then $\lim_{t \to 0} (T(t)f - f)/t = Lf$ in $L^p(\Omega, \mu)$ and consequently in $L^1(\Omega, \mu)$. Integrating we obtain

$$\int_{\Omega} Lf \, d\mu = \lim_{t \to 0} \frac{1}{t} \int_{\Omega} (T(t)f - f) \, d\mu = 0.$$

“⇐” Let $f \in D(L)$. Then the function $t \mapsto T(t)f$ belongs to $C^1([0, +\infty); L^p(\Omega, \mu))$ and $d/dt T(t)f = LT(t)f$, so that for every $t \geq 0$,

$$\frac{d}{dt} \int_X T(t)f \, d\mu = \int_X LT(t)f \, d\mu = 0.$$

Therefore the function $t \mapsto \int_X T(t)f \, d\mu$ is constant, and equal to $\int_X f \, d\mu$. The operator $L^p(\Omega, \mu) \to \mathbb{R}$, $f \mapsto \int_\Omega (T(t)f - f) \, d\mu$, is bounded and vanishes on the dense subset $D(L)$; hence it vanishes in the whole $L^p(\Omega, \mu)$. \hfill \Box

11.4 Analytic semigroups

We recall now an important class of semigroups, the analytic semigroups generated by sectorial operators. For the definition of sectorial operators we need that $X$ is a complex Banach space.

**Definition 11.4.1.** A linear operator $L : D(L) \subset X \to X$ is called sectorial if there are $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $M > 0$ such that

$$\begin{align*}
(i) \quad \rho(L) &\supset S_{\theta, \omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\
(ii) \quad \|R(\lambda, L)\|_{\mathcal{L}(X)} &\leq \frac{M}{|\lambda - \omega|} \quad \forall \lambda \in S_{\theta, \omega}.
\end{align*}$$

(11.4.1)
Lecture 11

Sectorial operators with dense domains are infinitesimal generators of semigroups with noteworthy smoothing properties. The proof of the following theorem may be found in [EN, Chapter 2], [L, Chapter 2].

**Theorem 11.4.2.** Let $L$ be a sectorial operator with dense domain. Then it is the infinitesimal generator of a semigroup $\{T(t) : t \geq 0\}$ that enjoys the following properties.

(i) $T(t)x \in D(L^k)$ for every $t > 0$, $x \in X$, $k \in \mathbb{N}$.

(ii) There are $M_0$, $M_1$, $M_2$, ... , such that

\[
\begin{align*}
(a) & \quad \|T(t)\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}, \ t > 0, \\
(b) & \quad \|t^k (L - \omega I)^k T(t)\|_{\mathcal{L}(X)} \leq M_k e^{\omega t}, \ t > 0,
\end{align*}
\]

where $\omega$ is the constant in (11.4.1).

(iii) The function $t \mapsto T(t)$ belongs to $C^\infty((0, +\infty); \mathcal{L}(X))$, and the equality

\[
\frac{d^k}{dt^k} T(t) = L^k T(t), \ t > 0,
\]

holds.

(iv) The function $t \mapsto T(t)$ has a $\mathcal{L}(X)$-valued holomorphic extension in a sector $S_{\beta,0}$ with $\beta > 0$.

The name “analytic semigroup” comes from property (iv). If $\Omega$ is an open set in $\mathbb{C}$, and $Y$ is a complex Banach space, a function $f : \Omega \to Y$ is called holomorphic if it is differentiable at every $z_0 \in \Omega$ in the usual complex sense, i.e. there exists the limit

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0).
\]

As in the scalar case, such functions are infinitely many times differentiable at every $z_0 \in \Omega$, and the Taylor series $\sum_{k=0}^{\infty} f^{(k)}(z_0)(z - z_0)^k / k!$ converges to $f(z)$ for every $z$ in a neighborhood of $z_0$.

We do not present the proof of this theorem, because in the case of Ornstein-Uhlenbeck semigroup that will be discussed in the next lectures we shall provide direct proofs of the relevant properties without relying on the above general results. A more general theory of analytic semigroups, not necessarily strongly continuous at $t = 0$, is available, see [L].

### 11.4.1 Self-adjoint operators in Hilbert spaces

If $X$ is a Hilbert space (inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$) then we can say more on semigroups and generators in connection to self-adjointness. Notice also that the dissipativity condition can be rephrased in the Hilbert space as follows. An operator $L : D(L) \to X$ is dissipative iff (see Exercise 11.5)

\[
\text{Re} \langle Lx, x \rangle \leq 0, \ \forall x \in D(L).
\]

Let us prove that any self-adjoint dissipative operator is sectorial.
Proposition 11.4.3. Let \( L : D(L) \subset X \to X \) be a self-adjoint dissipative operator. Then \( L \) is sectorial with \( \theta < \pi \) arbitrary and \( \omega = 0 \).

Proof. Let us first show that the spectrum of \( L \) is real. If \( \lambda = a + ib \in \mathbb{C} \), for every \( x \in D(L) \) we have

\[
\| (\lambda I - L)x \|^2 = (a^2 + b^2)\| x \|^2 - 2a \langle x, Lx \rangle + \| Lx \|^2 \geq b^2 \| x \|^2,
\]

hence if \( b \neq 0 \) then \( \lambda I - L \) is injective. Let us check that in this case it is also surjective, showing that its range is closed and dense in \( X \). Let \( (x_n) \subset D(L) \) be a sequence such that the sequence \((\lambda x_n - Lx_n)\) is convergent. From the inequality

\[
\| (\lambda I - L)(x_n - x_m) \|^2 \geq b^2 \| x_n - x_m \|^2, \quad n, m \in \mathbb{N},
\]

it follows that the sequence \((x_n)\) is a Cauchy sequence, hence \((Lx_n)\) as well. Therefore, there are \( x, y \in X \) such that \( x_n \to x \) and \( Lx_n \to y \). Since \( L \) is closed, \( x \in D(L) \) and \( Lx = y \), hence \( \lambda x_n - Lx_n \) converges to \( \lambda x - Lx \in \text{rg} (\lambda I - L) \) and the range of \( \lambda I - L \) is closed.

Let now \( y \) be orthogonal to the range of \((\lambda I - L)\). Then, for every \( x \in D(L) \) we have \( \langle y, \lambda x - Lx \rangle = 0 \), whence \( y \in D(L^*) = D(L) \) and \( \lambda y - L^*y = \lambda y - Ly = 0 \). As \( \lambda I - L \) injective, \( y = 0 \) follows. Therefore the range of \((\lambda I - L)\) is dense in \( X \).

From the dissipativity of \( L \) it follows that the spectrum of \( L \) is contained in \((-\infty, 0]\). Indeed, if \( \lambda > 0 \) then for every \( x \in D(L) \) we have, instead of (11.4.5),

\[
\| (\lambda I - L)x \|^2 = \lambda^2 \| x \|^2 - 2\lambda \langle x, Lx \rangle + \| Lx \|^2 \geq \lambda^2 \| x \|^2,
\]

and arguing as above we deduce \( \lambda \in \rho(L) \).

Let us now estimate \( \| R(\lambda, L) \| \), for \( \lambda = re^{i\theta} \), with \( r > 0 \), \(-\pi < \theta < \pi \). For \( x \in X \), set \( u = R(\lambda, L)x \). Multiplying the equality \( \lambda u - Lu = x \) by \( e^{-i\theta/2} \) and then taking the inner product with \( u \), we get

\[
re^{i\theta/2} \| u \|^2 - e^{-i\theta/2} \langle Lu, u \rangle = e^{-i\theta/2} \langle x, u \rangle,
\]

whence, taking the real part,

\[
\rho \cos(\theta/2) \| u \|^2 - \cos(\theta/2) \langle Lu, u \rangle = \text{Re}(e^{-i\theta/2} \langle x, u \rangle) \leq \| x \| \| u \|
\]

and then, as \( \cos(\theta/2) > 0 \), also

\[
\| u \| \leq \frac{\| x \|}{|\lambda| \cos(\theta/2)}.
\]

with \( \theta = \arg \lambda \).

\[ \square \]

Proposition 11.4.4. Let \( \{T(t) : t \geq 0\} \) be a \( C_0 \)-semigroup. The family of operators \( \{T^*(t) : t \geq 0\} \) is a \( C_0 \)-semigroup whose generator is \( L^* \).
Proof. The semigroup law is immediately checked. Let us prove the strong continuity. Possibly considering the rescaled semigroup $e^{-\omega t}T(t)$ with $M, \omega$ as in (11.1.1), see Remark 11.1.8, we may assume that $\|T(t)\|_{\mathcal{L}(X)} \leq M$ for every $t \geq 0$, without loss of generality, $\|T(t)\| = \|T(t)^*\| \leq 1$ (see Exercise 11.6). For $x \in X$ we have

\[
\|T(t)^*x - x\|^2 = \langle T(t)^*x - x, T(t)^*x - x \rangle \\
= \|T(t)^*x\|^2 + \|x\|^2 - \langle x, T(t)^*x \rangle - \langle T(t)^*x, x \rangle \\
\leq 2\|x\|^2 - \left(\langle x, T(t)^*x \rangle + \langle T(t)^*x, x \rangle \right) \\
= 2\|x\|^2 - \left(\langle T(t)x, x \rangle + \langle x, T(t)x \rangle \right)
\]

whence

\[
\limsup_{t \to 0} \|T(t)^*x - x\| = 0
\]

by the strong continuity of $T(t)$, and then $T(\cdot)^*x$ is continuous at 0. By Lemma 11.1.2, $t \mapsto T(t)^*x$ is continuous on $[0, \infty)$ and $\{T(t)^*: t \geq 0\}$ is a $C_0$-semigroup. Denoting by $A$ its generator, for $x \in D(L)$ and $y \in D(A)$ we have

\[
\langle Lx, y \rangle = \lim_{t \to 0} \langle (t^{-1}(T(t) - I)x, y \rangle = \lim_{t \to 0} \langle x, t^{-1}(T(t)^* - I)y \rangle = \langle x, Ay \rangle,
\]

so that $A \subset L^*$. Conversely, for $y \in D(L^*)$, $x \in D(L)$ we have

\[
\langle x, T(t)^*y - y \rangle = \langle T(t)x - x, y \rangle = \int_0^t \langle LT(s)x, y \rangle \, ds \\
= \int_0^t \langle T(s)x, L^*y \rangle \, ds = \int_0^t \langle x, T(s)^*L^*y \rangle \, ds.
\]

We deduce

\[
T(t)^*y - y = \int_0^t T(s)^*L^*y \, ds,
\]

whence, dividing by $t$ and letting $t \to 0$ we get $Ay = L^*y$ for every $y \in D(L^*)$ and consequently $L^* \subset A$. \hfill \Box

The following result is an immediate consequence of Proposition 11.4.4.

Corollary 11.4.5. The generator $L$ is self-adjoint if and only if $T(t)$ is self-adjoint for every $t > 0$.

11.5 Exercises

Exercise 11.1. Let $\mathbb{R}$ be endowed with the Lebesgue measure $\lambda_1$, and let $f : [a, b] \to X$ be a continuous function. Prove that it is Bochner integrable, that

\[
\int_a^b f(t) \, dt = \lim_{n \to \infty} \sum_{i=1}^n f(\tau_i) \frac{b-a}{n}
\]
for any choice of \( \tau_i \in \left[ a + \frac{(b-a)(i-1)}{n}, a + \frac{(b-a)i}{n} \right], i = 1, \ldots, n \) (the sums in this approximation are the usual Riemann sums in the real-valued case) and that, setting

\[
F(t) = \int_a^t f(s)ds, \quad a \leq t \leq b,
\]

the function \( F \) is continuously differentiable, with

\[
F'(t) = f(t), \quad a \leq t \leq b.
\]

**Exercise 11.2.** Prove that if \( u \in C([0, +\infty); D(L)) \cap C^1([0, +\infty); X) \) is a solution of problem (11.1.3), then for \( t > 0 \) the function \( v(s) = T(t-s)u(s) \) is continuously differentiable in \([0, t]\) and it verifies \( v'(s) = -T(t-s)Lu(s) + T(t-s)u'(s) = 0 \) for \( 0 \leq s \leq t \).

**Exercise 11.3.** Let \( L : D(L) \subset X \to X \) be a linear operator. Prove that if \( \rho(L) \neq \emptyset \) then \( L \) is closed.

**Exercise 11.4.** Prove the resolvent identity (11.2.3).

**Exercise 11.5.** Prove that in Hilbert spaces the dissipativity condition in Definition 11.2.3 is equivalent to (11.4.4).

**Exercise 11.6.** Let \( \{T(t) : t \geq 0\} \) be a bounded strongly continuous semigroup. Prove that the norm

\[
|x| := \sup_{t \geq 0} \|T(t)x\|
\]

is equivalent to \( \|\cdot\| \) and that \( T(t) \) is contractive on \((X, |\cdot|)\).

**Bibliography**


