13

The Ornstein-Uhlenbeck operator

In this lecture we study the infinitesimal generator of $T_p(t)$, for $p \in (1, +\infty)$. The strongest result is the characterisation of the domain of the generator $L_2$ of $T_2(t)$ as the Sobolev space $W^{2,2}(X, \gamma)$. A similar result holds for $p \in (1, +\infty) \setminus \{2\}$, but the proof is much more complicated and will not be given here.

13.1 The finite dimensional case

Here, $X = \mathbb{R}^d$ and $\gamma = \gamma_d$. We describe the infinitesimal generator $L_p$ of $T_p(t)$ in $L^p(\mathbb{R}^d, \gamma_d)$, for $p \in (1, +\infty)$, which is a suitable realisation of the Ornstein-Uhlenbeck differential operator

$$\mathcal{L} f(x) := \Delta f(x) - x \cdot \nabla f(x)$$

(13.1.1)

in $L^p(\mathbb{R}^d, \gamma_d)$.

We recall that

$$D(L_p) = \left\{ f \in L^p(\mathbb{R}^d, \gamma_d) : \exists L^p - \lim_{t \to 0^+} \frac{T(t)f - f}{t} \right\},$$

$$L_p f = \lim_{t \to 0^+} \frac{T(t)f - f}{t}.$$

If $f \in D(L_p)$, by Lemma 11.1.7 the function $t \mapsto T(t)f$ belongs to $C^1([0, +\infty); L^p(\mathbb{R}^d, \gamma_d)) \cap C([0, +\infty); D(L_p))$ and $d/dt T(t)f = L_p T(t)f$, for every $t \geq 0$. To find an expression of $L_p$, we differentiate $T(t)f$ with respect to time for good $f$. We recall that for $f \in C_b(\mathbb{R}^d)$, $T_p(t)f = T(t)f$ is given by formula (12.1.1).

Lemma 13.1.1. For every $f \in C_b(\mathbb{R}^d)$, the function $(t,x) \mapsto T(t)f(x)$ is smooth in $(0, +\infty) \times \mathbb{R}^d$, and we have

$$\frac{d}{dt}(T(t)f)(x) = \Delta T(t)f(x) - x \cdot \nabla T(t)f(x), \quad t > 0, \quad x \in \mathbb{R}^d.$$
If \( f \in C^2_b(\mathbb{R}^d) \), for every \( x \in \mathbb{R}^d \) the function \( t \mapsto T(t)f(x) \) is differentiable also at \( t = 0 \), with
\[
\frac{d}{dt}(T(t)f)(x)|_{t=0} = \Delta f(x) - x \cdot \nabla f(x), \quad x \in \mathbb{R}^d, \tag{13.1.3}
\]
and the function \( (t,x) \mapsto d/dt(T(t)f)(x) \) is continuous in \([0, +\infty) \times \mathbb{R}^d\).

Proof. Setting \( z = e^{-t}x + \sqrt{1 - e^{-2t}} y \) in (12.1.1) we see that \( (t,x) \mapsto T(t)f(x) \) is smooth in \((0, +\infty) \times \mathbb{R}^d\), and that
\[
\frac{d}{dt}(T(t)f)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(z) \frac{\partial}{\partial t} \left( \exp \left\{ -\frac{|z - e^{-t}x|^2}{2(1 - e^{-2t})} \right\} (1 - e^{-2t})^{-d/2} \right) dz
\]
\[
= \frac{(1 - e^{-2t})^{-d/2}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(z) \exp \left\{ -\frac{|z - e^{-t}x|^2}{2(1 - e^{-2t})} \right\} \left( 1 - e^{-2t} \right)^{-d/2} \left( \frac{e^{-t}(z - e^{-t}x) \cdot x}{1 - e^{-2t}} + \frac{e^{-2t}|z - e^{-t}x|^2}{(1 - e^{-2t})^2} - \frac{de^{-2t}}{1 - e^{-2t}} \right) dz
\]
\[
= \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1 - e^{-2t}} y)(-c(t) y \cdot x + c(t)|y|^2 - de(t)^2) \gamma_d(dy),
\]
where \( c(t) = e^{-t}/\sqrt{1 - e^{-2t}} \). Differentiating twice with respect to \( x \) in (12.1.1) (recall (12.1.5)), we obtain
\[
D_{ij}(T(t)f)(x) = c(t)^2 \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1 - e^{-2t}} y)(-\delta_{ij} + y_i y_j) \gamma_d(dy)
\]
so that
\[
\Delta T(t)f(x) = c(t)^2 \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1 - e^{-2t}} y)(-d + |y|^2) \gamma_d(dy).
\]
Therefore,
\[
\frac{d}{dt}(T(t)f)(x) - \Delta T(t)f(x) = -c(t) \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1 - e^{-2t}} y) x \cdot y \gamma_d(dy)
\]
\[
= -\nabla T(t)f(x) \cdot x,
\]
and (13.1.2) follows.

For \( f \in C^2_b(\mathbb{R}^d) \), we rewrite formula (13.1.2) as
\[
\frac{d}{dt}(T(t)f)(x) = (\mathcal{L}T(t)f)(x) \tag{13.1.4}
\]
\[
= e^{-2t}(T(t)\Delta f)(x) - e^{-t}x \cdot (T(t)\nabla f)(x), \quad t > 0, \quad x \in \mathbb{R}^d,
\]
taking into account (12.1.7) and (12.1.9). (We recall that \( (T(t)\nabla f)(x) \) is the vector whose \( j \)-th component is \( T(t)D_j f(x) \)). Since \( \Delta f \) and each \( D_j f \) are continuous and bounded in \( \mathbb{R}^d \), the right hand side is continuous in \([0, +\infty) \times \mathbb{R}^d\). So, for every \( x \in \mathbb{R}^d \) the function \( \theta(t) := T(t)f(x) \) is continuous in \([0, +\infty) \), it is differentiable in \((0, +\infty) \) and \( \lim_{t \to 0} \theta'(t) = \Delta f(x) - x \cdot \nabla f(x) \). Therefore, \( \theta \) is differentiable at 0 too, (13.1.4) holds also at \( t = 0 \), and (13.1.3) follows.
Lemma 13.1.1 suggests that $L_p$ is a suitable realisation of the Ornstein-Uhlenbeck differential operator $\mathcal{L}$ defined in (13.1.1). For a first characterisation of $L_p$, we use Lemma 11.1.9.

**Proposition 13.1.2.** For $1 \leq p < \infty$ and $k \in \mathbb{N}$, $k \geq 2$, the operator $\mathcal{L} : D(\mathcal{L}) = C^k_b(\mathbb{R}^d) \subset L^p(\mathbb{R}^d, \gamma_d) \to L^p(\mathbb{R}^d, \gamma_d)$ is closable, and its closure is $L_p$. So, $D(L_p)$ consists of all $f \in L^p(\mathbb{R}^d, \gamma_d)$ for which there exists a sequence of functions $(f_n) \subset C^k_b(\mathbb{R}^d)$ such that $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$ and $(\mathcal{L} f_n)$ converges in $L^p(\mathbb{R}^d, \gamma_d)$. In this case, $L_p f = \lim_{n \to \infty} \mathcal{L} f_n$.

**Proof.** We check that $D = C^k_b(\mathbb{R}^d)$ satisfies the assumptions of Proposition 11.1.9, i.e., it is a core of $L_p$. We already know, from Proposition 12.1.4, that $T(t)$ maps $C^k_b(\mathbb{R}^d)$ into itself for $k = 1, 2$. The proof of the fact that $C^k_b(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, \gamma_d)$ and that $T(t)$ maps $C^k_b(\mathbb{R}^d)$ into itself for $k \geq 3$ is left as Exercise 13.2.

Since $C^k_b(\mathbb{R}^d) \subset C^2_b(\mathbb{R}^d)$ for $k \geq 2$, it remains to prove that $C^2_b(\mathbb{R}^d) \subset D(L_p)$, and $L_p f = \mathcal{L} f$ for every $f \in C^2_b(\mathbb{R}^d)$.

By Lemma 13.1.1, for every $f \in C^2_b(\mathbb{R}^d)$ we have $d/dt T(t)f(x) = \mathcal{L} T(t)f(x)$ for every $x \in \mathbb{R}^d$ and $t \geq 0$; moreover $t \mapsto d/dt T(t)f(x)$ is continuous for every $x$. Therefore for every $t > 0$ we have

$$\frac{T(t)f(x) - f(x)}{t} = \frac{1}{t} \int_0^t d \frac{d}{ds} T(s)f(x) ds = \frac{1}{t} \int_0^t \mathcal{L} T(s)f(x) ds,$$

and

$$\int_{\mathbb{R}^d} \left| \frac{T(t)f(x) - f(x)}{t} - \mathcal{L} f(x) \right|^p \gamma_d(dx) \leq \int_{\mathbb{R}^d} \left( \frac{1}{t} \int_0^t |\mathcal{L} T(s)f(x) - \mathcal{L} f(x)| ds \right)^p \gamma_d(dx).$$

Since $s \mapsto \mathcal{L} T(s)f(x)$ is continuous, for every $x$ we have

$$\lim_{t \to 0^+} \left( \frac{1}{t} \int_0^t |\mathcal{L} T(s)f(x) - \mathcal{L} f(x)| ds \right)^p = 0.$$

Moreover, by (13.1.4),

$$\left( \frac{1}{t} \int_0^t |\mathcal{L} T(s)f(x) - \mathcal{L} f(x)| ds \right)^p \leq 2^p (\|\Delta f\|_\infty + \|x\| \|\nabla f\|_\infty)^p \in L^1(\mathbb{R}^d, \gamma_d).$$

By the Dominated Convergence Theorem,

$$\lim_{t \to 0^+} \int_{\mathbb{R}^d} \left| \frac{T(t)f(x) - f(x)}{t} - \mathcal{L} f(x) \right|^p \gamma_d(dx) = 0.$$

Then, $C^k_b(\mathbb{R}^d) \subset D(L_p)$. Since $L_p$ is a closed operator and it is an extension of $\mathcal{L} : C^k_b(\mathbb{R}^d) \to L^p(\mathbb{R}^d, \gamma_d)$, $\mathcal{L} : C^k_b(\mathbb{R}^d) \to L^p(\mathbb{R}^d, \gamma_d)$ is closable.

Applying Lemma 11.1.9 with $D = C^k_b(\mathbb{R}^d)$, we obtain that $D(L_p)$ is the closure of $C^k_b(\mathbb{R}^d)$ in the graph norm of $L_p$, namely $f \in D(L_p)$ iff there exists a sequence $(f_n) \subset C^k_b(\mathbb{R}^d)$ such that $f_n \to f$ in $L^p(\mathbb{R}^d, \gamma_d)$ and $L_p f_n = \mathcal{L} f_n$ converges in $L^p(\mathbb{R}^d, \gamma_d)$. This shows that $L_p$ is the closure of $\mathcal{L} : C^k_b(\mathbb{R}^d) \to L^p(\mathbb{R}^d, \gamma_d)$. \qed
In the case \( p = 2 \) we obtain other characterisations of \( D(L_2) \). To start with, we point out some important properties of \( \mathcal{L} \), when applied to elements of \( W^{2,2}(\mathbb{R}^d, \gamma_d) \).

**Lemma 13.1.3.**  
(a) \( \mathcal{L} : W^{2,2}(\mathbb{R}^d, \gamma_d) \to L^2(\mathbb{R}^d, \gamma_d) \) is a bounded operator;  
(b) for every \( f \in W^{2,2}(\mathbb{R}^d, \gamma_d) \), \( g \in W^{1,2}(\mathbb{R}^d, \gamma_d) \) we have  
\[
\int_{\mathbb{R}^d} \mathcal{L} f g \, d\gamma_d = - \int_{\mathbb{R}^d} \nabla f \cdot \nabla g \, d\gamma_d.  \tag{13.1.5}
\]
(c) for every \( f \in W^{2,2}(\mathbb{R}^d, \gamma_d) \) we have  
\[
\mathcal{L} f = \text{div}_{\gamma_d} \nabla f.  \tag{13.1.6}
\]

**Proof.** To prove (a) it is sufficient to show that the mapping \( T : W^{2,2}(\mathbb{R}^d, \gamma_d) \to L^2(\mathbb{R}^d, \gamma_d) \) defined by \( (T f)(x) := x \cdot \nabla f(x) \) is bounded. For every \( i = 1, \ldots, d \), set \( g_i(x) = x_i D_i f(x) \). The mapping \( f \mapsto g_i \) is bounded from \( W^{2,2}(\mathbb{R}^d, \gamma_d) \) to \( L^2(\mathbb{R}^d, \gamma_d) \) by Lemma 10.2.6, and summing up the statement follows.

To prove (b) it is sufficient to apply the integration by parts formula (9.1.3) to compute \( \int_{\mathbb{R}^d} D_i f g \, d\gamma_d \), for every \( i = 1, \ldots, d \), and to sum up. In fact, (9.1.3) was stated for \( C^1 \) functions, but it is readily extended to Sobolev functions using Proposition 9.1.5.

Statement (c) follows from Theorem 10.2.7. In this case we have \( H = \mathbb{R}^d \), and it is convenient to take the canonical basis of \( \mathbb{R}^d \) as a basis for \( H \). So, we have \( h_i(x) = x_i \) for \( i = 1, \ldots, d \) and \( \text{div}_{\gamma_d} v(x) = \sum_{i=1}^d D_i v_i - x_i v_i \), for every \( v \in W^{1,2}(\mathbb{R}^d, \gamma_d; \mathbb{R}^d) \). Taking \( v = \nabla f \), (13.1.6) follows. \( \square \)

The first characterisation of \( D(L_2) \) is the following.

**Theorem 13.1.4.** \( D(L_2) = W^{2,2}(\mathbb{R}^d, \gamma_d) \), and \( L_2 f = \mathcal{L} f \) for every \( f \in W^{2,2}(\mathbb{R}^d, \gamma_d) \). Moreover, for every \( f \in W^{2,2}(\mathbb{R}^d, \gamma_d) \),  
\[
\| f \|_{L^2(\mathbb{R}^d; \gamma_d)} + \| \mathcal{L} f \|_{L^2(\mathbb{R}^d; \gamma_d)} \leq \| f \|_{W^{2,2}(\mathbb{R}^d; \gamma_d)} \leq \frac{3}{2}(\| f \|_{L^2(\mathbb{R}^d; \gamma_d)} + \| \mathcal{L} f \|_{L^2(\mathbb{R}^d; \gamma_d)}).  \tag{13.1.7}
\]

**Proof.** The embedding \( W^{2,2}(\mathbb{R}^d, \gamma_d) \subset D(L_2) \) is an easy consequence of Lemma 13.1.3(a). For every \( f \in W^{2,2}(\mathbb{R}^d, \gamma_d) \) there is a sequence \( (f_k) \) of \( C^2_b \) functions such that \( f_k \to f \) in \( W^{2,2}(\mathbb{R}^d, \gamma_d) \). Then, \( \mathcal{L} f_k \to g = \mathcal{L} f \) in \( L^2(\mathbb{R}^d, \gamma_d) \). By Proposition 13.1.2, \( f \in D(L_2) \) and \( L_2 f = \mathcal{L} f \) for every \( f \in W^{2,2}(\mathbb{R}^d, \gamma_d) \). However, the embedding constant that comes from Lemma 13.1.3(a) is not clear, and may depend on \( d \). It is better to use (13.1.6) and Theorem 10.2.7, with \( v = \nabla f \), that gives the clean estimate  
\[
\| \mathcal{L} f \|_{L^2(\mathbb{R}^d; \gamma_d)} \leq \| \nabla f \|_{W^{1,2}(\mathbb{R}^d; \gamma_d; \mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \sum_{i=1}^d (D_i f)^2 \, d\gamma_d \right)^{1/2} + \left( \int_{\mathbb{R}^d} \sum_{i,j=1}^d (D_{ij} f)^2 \, d\gamma_d \right)^{1/2}
\]

which yields  
\[
\| f \|_{L^2(\mathbb{R}^d; \gamma_d)} + \| \mathcal{L} f \|_{L^2(\mathbb{R}^d; \gamma_d)} \leq \| f \|_{W^{2,2}(\mathbb{R}^d, \gamma_d)}, \quad f \in W^{2,2}(\mathbb{R}^d, \gamma_d).
\]
To prove the other embedding we shall show that
\[
\|f\|_{W^{2,2}(\mathbb{R}^d, \gamma_d)} \leq \frac{3}{2} (\|f\|_{L^2(\mathbb{R}^d, \gamma_d)} + \|\mathcal{L}f\|_{L^2(\mathbb{R}^d, \gamma_d)})
\] (13.1.8)
for every \(f \in C^3_b(\mathbb{R}^d)\). Indeed, by Proposition 13.1.2, if \(f \in D(L_2)\) there is a sequence \((f_k)\) of \(C^3_b\) functions such that \(f_k\) converges to \(f\) and \(\mathcal{L}f_k\) converges in \(L^2(\mathbb{R}^d, \gamma_d)\). Applying estimate (13.1.8) to \(f_k - f_h\) we obtain that \((f_k)\) is a Cauchy sequence in \(W^{2,2}(\mathbb{R}^d, \gamma_d)\), so that its \(L^2\) limit \(f\) belongs to \(W^{2,2}(\mathbb{R}^d, \gamma_d)\), and \(f\) satisfies estimate (13.1.8), too. This shows that \(D(L_2)\) is continuously embedded in \(W^{2,2}(\mathbb{R}^d, \gamma_d)\) and that (13.1.8) holds for every \(f \in D(L_2)\).

So, let us prove that (13.1.8) holds for every \(f \in C^3_b(\mathbb{R}^d)\). By Lemma 13.1.3(b),
\[
\int_{\mathbb{R}^d} |\nabla f|^2 d\gamma_d = -\int_{\mathbb{R}^d} f\mathcal{L}f d\gamma_d \leq \|f\|_{L^2(\mathbb{R}^d, \gamma_d)} \|\mathcal{L}f\|_{L^2(\mathbb{R}^d, \gamma_d)}.
\] (13.1.9)
To estimate the \(L^2\) norm of the second order derivatives, we set \(\mathcal{L}f =: g\) and we differentiate with respect to \(x_j\) (this is why we consider \(C^3_b\), instead of only \(C^2_b\), functions) for every \(j = 1, \ldots, d\). We obtain
\[
D_j(\Delta f) - \sum_{i=1}^d (\delta_{ij} D_i f + x_i D_{ji} f) = D_j g.
\]
Multiplying by \(D_j f\) and summing up we get
\[
\sum_{j=1}^d D_j f \Delta(D_j f) - |\nabla f|^2 - \sum_{j=1}^d x \cdot \nabla(D_j f) D_j f = \nabla f \cdot \nabla g.
\]
Note that each term in the above sum belongs to \(L^p(\mathbb{R}^d, \gamma_d)\) for every \(p > 1\). We integrate over \(\mathbb{R}^d\) and we obtain
\[
\int_{\mathbb{R}^d} \left( \sum_{j=1}^d D_j f \mathcal{L}(D_j f) - |\nabla f|^2 \right) d\gamma_d = \int_{\mathbb{R}^d} \nabla f \cdot \nabla g \, d\gamma_d.
\]
Now we use the integration formula (13.1.5), both in the left hand side and in the right hand side, obtaining
\[
-\int_{\mathbb{R}^d} \sum_{j=1}^d |\nabla D_j f|^2 d\gamma_d - \int_{\mathbb{R}^d} |\nabla f|^2 d\gamma_d = -\int_{\mathbb{R}^d} g \mathcal{L} f \, d\gamma_d
\]
so that, since \(g = \mathcal{L} f\)
\[
\int_{\mathbb{R}^d} \sum_{i,j=1}^d (D_{ij} f)^2 d\gamma_d = \int_{\mathbb{R}^d} (\mathcal{L} f)^2 d\gamma_d - \int_{\mathbb{R}^d} |\nabla f|^2 d\gamma_d \leq \int_{\mathbb{R}^d} (\mathcal{L} f)^2 d\gamma_d.
\]
This inequality and (13.1.9) yield (13.1.8). Indeed,
\[
\|f\|_{W^{2,2}} \leq \|f\|_{L^2} + (\|f\|_{L^2}\|\mathcal{L}f\|_{L^2})^{1/2} + \|\mathcal{L}f\|_{L^2} \\
\leq \|f\|_{L^2} + \frac{1}{2}(\|f\|_{L^2} + \|\mathcal{L}f\|_{L^2}) + \|\mathcal{L}f\|_{L^2} \\
\leq \frac{3}{2}(\|f\|_{L^2} + \|\mathcal{L}f\|_{L^2}).
\]

The next characterisation fits last year Isem. We recall below general results about bilinear forms in Hilbert spaces. We only need a basic result; more refined results are in last year Isem lecture notes.

Let $V \subset W$ be real Hilbert spaces, with continuous and dense embedding, and let $\mathcal{Q}: V \times V \to \mathbb{R}$ be a bounded bilinear form. “Bounded” means that there exists $M > 0$ such that $|\mathcal{Q}(u, v)| \leq M\|u\|_V\|v\|_V$ for every $u, v \in V$; “bilinear” means that $\mathcal{Q}$ is linear both with respect to $u$ and with respect to $v$. $\mathcal{Q}$ is called “nonnegative” if $\mathcal{Q}(u, u) \geq 0$ for every $u \in V$, and “coercive” if there is $c > 0$ such that $\mathcal{Q}(u, u) \geq c\|u\|_V^2$, for every $u \in V$; it is called “symmetric” if $\mathcal{Q}(u, v) = \mathcal{Q}(v, u)$ for every $u, v \in V$. Note that the form in (13.1.10), with $V = W^{1,2}(\mathbb{R}^d, \gamma_d), W = L^2(\mathbb{R}^d, \gamma_d)$ is bounded, bilinear, symmetric and nonnegative. It is not coercive, but $\mathcal{Q}(u, v) + \alpha \langle u, v \rangle_{L^2(\mathbb{R}^d, \gamma_d)}$ is coercive for every $\alpha > 0$.

For any bounded bilinear form $\mathcal{Q}$, an unbounded linear operator $A$ in the space $W$ is naturally associated with $\mathcal{Q}$. $D(A)$ consists of the elements $u \in V$ such that the mapping $V \to \mathbb{R}$, $v \mapsto \mathcal{Q}(u, v)$, has a linear bounded extension to the whole $W$. By the Riesz Theorem, this is equivalent to the existence of $g \in W$ such that $\mathcal{Q}(u, v) = \langle g, v \rangle_W$, for every $v \in V$. Note that $g$ is unique, because $V$ is dense in $W$. Then we set $Au = -g$, where $g$ is the unique element of $W$ such that $\mathcal{Q}(u, v) = \langle g, v \rangle_W$, for every $v \in V$.

**Theorem 13.1.5.** Let $V \subset W$ be real Hilbert spaces, with continuous and dense embedding, and let $\mathcal{Q}: V \times V \to \mathbb{R}$ be a bounded bilinear symmetric form, such that $(u, v) \mapsto \mathcal{Q}(u, v) + \alpha \langle u, v \rangle_W$ is coercive for some $\alpha > 0$. Then the operator $A : D(A) \to W$ defined above is densely defined and self-adjoint. If in addition $\mathcal{Q}$ is nonnegative, $A$ is dissipative.

*Proof.* The mapping $(u, v) \mapsto \mathcal{Q}(u, v) + \alpha \langle u, v \rangle_W$ is an inner product in $V$, and the associated norm is equivalent to the $V$-norm, by the continuity of $\mathcal{Q}$ and the coercivity assumption.

It is convenient to consider the operator $\tilde{A} : D(\tilde{A}) = D(A) \to W$, $\tilde{A}u := Au + \alpha u$. Of course if $\tilde{A}$ is self-adjoint, also $A$ is self-adjoint.

We consider the canonical isomorphism $T : V \to V^*$ defined by $(Tu)(v) = \mathcal{Q}(u, v) + \alpha \langle u, v \rangle_W$ (we are using the new inner product above defined), and the embedding $J : W \to V^*$, such that $(Ju)(v) = \langle u, v \rangle_W$. $T$ is an isometry by the Riesz Theorem, and $J$ is bounded since for every $u \in W$ and $v \in V$ we have $|(Ju)(v)| \leq \|u\|_W\|v\|_W \leq C\|u\|_W\|v\|_V$, where $C$ is the norm of the embedding $V \subset W$. Moreover, $J$ is one to one, since $V$ is dense in $W$. 


By definition, \( u \in D(\tilde{A}) \) iff there exists \( g \in W \) such that \( \mathcal{Q}(u, v) + \alpha \langle u, v \rangle_W = \langle g, v \rangle_W \) for every \( v \in V \), which means \( Tu = Jg \), and in this case \( \tilde{A}u = -g \).

The range of \( J \) is dense in \( V^* \). If it were not, there would exist \( \Phi \in V^* \setminus \{0\} \) such that \( \langle Ju, \Phi \rangle = 0 \) for every \( w \in W \). So, there would exists \( \varphi \in V \setminus \{0\} \) such that \( Ju(\varphi) = 0 \), namely \( \langle w, \varphi \rangle_W = 0 \) for every \( w \in W \). This implies \( \varphi = 0 \), a contradiction. Since \( T \) is an isomorphism, the range of \( T^{-1}J \), which is nothing but the domain of \( \tilde{A} \), is dense in \( V \). Since \( V \) is in its turn dense in \( W \), then \( D(\tilde{A}) \) is dense in \( W \).

The symmetry of \( \mathcal{Q} \) implies immediately that \( \tilde{A} \) is self–adjoint. Indeed, for \( u, v \in D(\tilde{A}) \) we have

\[
\langle \tilde{A}u, v \rangle_W = \mathcal{Q}(u, v) + \alpha \langle u, v \rangle_W = \mathcal{Q}(v, u) + \alpha \langle v, u \rangle_W = \langle u, \tilde{A}v \rangle_W.
\]

Since \( \tilde{A} \) is onto, it is self–adjoint.

The last statement is obvious: since \( \langle Au, u \rangle = -\mathcal{Q}(u, u) \) for every \( u \in D(A) \), if \( \mathcal{Q} \) is nonnegative then \( A \) is dissipative. 

In our setting the bilinear form is

\[
\mathcal{Q}(u, v) := \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, d\gamma_d, \quad u, v \in W^{1,2}(\mathbb{R}^d, \gamma_d), \tag{13.1.10}
\]

so that the assumptions of Theorem 13.1.5 are satisfied with \( W = L^2(\mathbb{R}^d, \gamma_d) \), \( V = W^{1,2}(\mathbb{R}^d, \gamma_d) \) and every \( \alpha > 0 \). \( D(A) \) is the set

\[
\left\{ u \in W^{1,2}(\mathbb{R}^d, \gamma_d) : \exists g \in L^2(\mathbb{R}^d, \gamma_d) \text{ such that } \mathcal{Q}(u, v) = \int_{\mathbb{R}^d} g \, v \, d\gamma_d, \forall v \in W^{1,2}(\mathbb{R}^d, \gamma_d) \right\}
\]

and \( Au = -g \).

**Theorem 13.1.6.** Let \( \mathcal{Q} \) be the bilinear form in (13.1.10). Then \( D(A) = W^{2,2}(\mathbb{R}^d, \gamma_d) \), and \( A = L_2 \).

**Proof.** Let \( u \in W^{2,2}(\mathbb{R}^d, \gamma_d) \). By (13.1.5) and Theorem 13.1.4, for every \( v \in W^{1,2}(\mathbb{R}^d, \gamma_d) \) we have

\[
\mathcal{Q}(u, v) = -\int_{\mathbb{R}^d} \mathcal{L} u \, v \, d\gamma_d
\]

Therefore, the function \( g = \mathcal{L} u = L_2 u \) fits the definition of \( Au \) (recall that \( g \in L^2(\mathbb{R}^d, \gamma_d) \) by Lemma 13.1.3(a)). So, \( W^{2,2}(\mathbb{R}^d, \gamma_d) \subset D(A) \) and \( Au = L_2 u \) for every \( u \in W^{2,2}(\mathbb{R}^d, \gamma_d) \) (the last equality follows from Theorem 13.1.4). In other words, \( A \) is a self–adjoint extension of \( L_2 \). \( L_2 \) itself is self–adjoint by Corollary 11.4.5, because \( T_2(t) \) is self–adjoint in \( L^2(\mathbb{R}^d, \gamma_d) \) by Proposition 12.1.5(ii), for every \( t > 0 \). Self–adjoint operators have no proper self–adjoint extensions (this is because \( D(L_2) \subset D(A) \Rightarrow D(A^*) \subset D(L_2) \), but \( D(A^*) = D(A) \) and \( D(L_2^2) = D(L_2) \), hence \( D(A) = D(L_2) \) and \( A = L_2 \).
13.2 The infinite dimensional case

Here, as usual, $X$ is a separable Banach space endowed with a centred nondegenerate Gaussian measure $\gamma$, and $H$ is the relevant Cameron-Martin space.

The connection between finite dimension and infinite dimension is provided by the cylindrical functions. In the next proposition we show that suitable cylindrical functions belong to $D(L_p)$ for every $p \in (1, +\infty)$, and we write down an explicit expression of $L_p f$ for such $f$. Precisely, we fix an orthonormal basis $\{h_j : j \in \mathbb{N}\}$ of $H$ contained in $R_\gamma(X^*)$, and we denote by $\Sigma$ the set of the cylindrical functions of the type $f(x) = \varphi(h_1(x), \ldots, h_d(x))$ with $\varphi \in C_b^2(\mathbb{R}^d)$, for some $d \in \mathbb{N}$. This is a dense subspace of $L^p(X, \gamma)$ for every $p \in [1, +\infty)$, see Exercise 13.3. For such $f$, we have

$$\partial_i f(x) = \frac{\partial \varphi}{\partial \xi_i}(h_1(x), \ldots, h_d(x)), \ i \leq d; \ \ \ \ \partial_i f(x) = 0, \ i > d. \ \ (13.2.1)$$

To distinguish between the finite and the infinite dimensional case, we use the superscript $(d)$ when dealing with the Ornstein-Uhlenbeck semigroup and the Ornstein-Uhlenbeck semigroup in $\mathbb{R}^d$. So, $L_p^{(d)}$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $T^{(d)}(t)$ in $L^p(\mathbb{R}^d, \gamma_d)$. We recall that $L_p^{(d)}$ is a realisation of the operator $L^{(d)} = \Delta - x \cdot \nabla$, namely $L_p^{(d)} f = L^{(d)} f$ for every $f \in D(L_p^{(d)})$.

**Proposition 13.2.1.** Let $\{h_j : j \in \mathbb{N}\}$ be any orthonormal basis of $H$ contained in $R_\gamma(X^*)$, and let $f(x) = \varphi(h_1(x), \ldots, h_d(x))$ with $\varphi \in L^p(\mathbb{R}^d, \gamma_d)$, for some $d \in \mathbb{N}$ and $p \in [1, +\infty)$. Then for every $t > 0$ and $\gamma$-a.e. $x \in X$,

$$T_p(t)f(x) = (T_p^{(d)}(t)\varphi)(h_1(x), \ldots, h_d(x)).$$

If in addition $\varphi \in D(L_p^{(d)})$, then $f \in D(L_p)$, and

$$L_p f(x) = L_p^{(d)} \varphi(h_1(x), \ldots, h_d(x)).$$

If $\varphi \in C_b^2(\mathbb{R}^d)$, then

$$L_p f(x) = L^{(d)} \varphi(h_1(x), \ldots, h_d(x)) = \sum_{i=1}^d (\partial_i f(x) - h_i(x) \partial_i f(x)) = \text{div}_\gamma \nabla_H f(x).$$

**Proof.** Assume first that $\varphi \in C_b(\mathbb{R}^d)$. For $t > 0$ we have

$$T(t)f(x) = \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \gamma(dy)$$

$$= \int_X \varphi(e^{-t}h_1(x) + \sqrt{1 - e^{-2t}}h_1(y), \ldots, e^{-t}h_d(x) + \sqrt{1 - e^{-2t}}h_d(y)) \gamma(dy)$$

$$= \int_{\mathbb{R}^d} \varphi(e^{-t}h_1(x) + \sqrt{1 - e^{-2t}}\xi_1, \ldots, e^{-t}h_d(x) + \sqrt{1 - e^{-2t}}\xi_d) \gamma_d(d\xi)$$

$$= (T^{(d)}(t)\varphi)(h_1(x), \ldots, h_d(x)).$$
because $\gamma \circ (\hat{h}_1, \ldots, \hat{h}_d)^{-1} = \gamma_d$ by Exercise 2.4. If $\varphi \in L^p(\mathbb{R}^d, \gamma_d)$ is not continuous, we approximate it in $L^p(\mathbb{R}^d, \gamma_d)$ by a sequence of continuous and bounded functions $\varphi_n$. The sequence $f_n(x) := \varphi_n(h_1(x), \ldots, h_d(x))$ converges to $f$ and the sequence $g_n(x) := (T^{(d)}(t)\varphi_n)(h_1(x), \ldots, h_d(x))$ converges to $(T^{(d)}(t)\varphi)(h_1(x), \ldots, h_d(x))$ in $L^p(X, \gamma)$, still by Exercise 2.4. Therefore, $T(t)f_n$ converges to $T(t)f$ in $L^p(X, \gamma)$ for every $t > 0$, and the first statement follows.

Let now $\varphi \in D(L^{(d)}_p)$. For every $t > 0$ we have

$$\int_X \frac{|T(t)f(x) - f(x)|}{t} L^{(d)}_p(\hat{h}_1, \ldots, \hat{h}_d) \left| \gamma(dx) \right| = \int_X \left| \frac{T^{(d)}(t)\varphi(h_1(x), \ldots, h_d(x)) - \varphi(h_1(x), \ldots, h_d(x))}{t} - L^{(d)}_p(\hat{h}_1, \ldots, \hat{h}_d) \right| \left| \varphi(dx) \right|$$

that vanishes as $t \to 0$. So, the second statement follows.

Let $\varphi \in C^2_b(\mathbb{R}^d)$. By Theorem 13.1.2 we have

$$L^{(d)}_p(\varphi(x)) = \sum_{i=1}^d (D_{ii}\varphi(x) - \xi_i D_i \varphi(x)) = \mathcal{L}^{(d)}(\varphi(x)) = L^{(d)}_p(\varphi(x)), \quad \xi \in \mathbb{R}^d.$$  

Therefore,

$$L_p f(x) = (\mathcal{L}^{(d)}(\varphi)(\hat{h}_1, \ldots, \hat{h}_d(x)) = \sum_{i=1}^d (D_{ii}\varphi(x) - \xi_i D_i \varphi(x))\mid_{\xi = (h_1(x), \ldots, h_d(x))}$$

$$= \sum_{i=1}^d (\partial_{ii} f(x) - \hat{h}_i(x) \partial_i f(x)),$$

which coincides with $\text{div}_\gamma \nabla_H f(x)$. See Theorem 10.2.7. \hfill \Box

As a consequence of Propositions 13.2.1 and 11.1.9, we obtain a characterisation of $D(L_p)$ which is quite similar to the finite dimensional one.

**Theorem 13.2.2.** Let $\{h_j : j \in \mathbb{N}\}$ be an orthonormal basis of $H$ contained in $R_{\gamma}(X^*)$. Then the subspace $\Sigma$ of $C^2_b(X)$ defined above is a core of $L_p$ for every $p \in [1, +\infty)$, the restriction of $L_p$ to $\Sigma$ is closable in $L^p(X, \gamma)$ and its closure is $L_p$. In other words, $D(L_p)$ consists of all $f \in L^p(X, \gamma)$ such that there exists a sequence $(f_n)$ in $\Sigma$ which converges to $f$ in $L^p(X, \gamma)$ and such that $L_p f_n = \text{div}_\gamma \nabla_H f_n$ converges in $L^p(X, \gamma)$.

**Proof.** By Proposition 13.2.1, $\Sigma \subset D(L_p)$. For every $t > 0$, $T(t)f \in \Sigma$ if $f \in \Sigma$, by Proposition 13.2.1 and Proposition 12.1.4. By Lemma 11.1.9, $\Sigma$ is a core of $L_p$. \hfill \Box

For $p = 2$ we can prove other characterisations.
Theorem 13.2.3. \( D(L_2) = W^{2,2}(X, \gamma) \), and for every \( f \in W^{2,2}(X, \gamma) \) we have
\[
L_2 f = \text{div}_E \nabla_H f,
\]
and
\[
\|f\|_{L^2(X, \gamma)} + \|L_2 f\|_{L^2(X, \gamma)} \leq \|f\|_{W^{2,2}(X, \gamma)} \leq \frac{3}{2}(\|f\|_{L^2(X, \gamma)} + \|L_2 f\|_{L^2(X, \gamma)}).
\]

Proof. Fix an orthonormal basis of \( H \) contained in \( R_\gamma(X^*) \). By Exercise 13.3, \( \Sigma \) is dense in \( W^{2,2}(X, \gamma) \), and by Theorem 13.2.2 it is dense in \( D(L_2) \).

We claim that every \( f \in \Sigma \) satisfies (13.2.3), so that the \( W^{2,2} \) norm is equivalent to the graph norm of \( L_2 \) on \( \Sigma \). For every \( f \in \Sigma \), if \( f(x) = \varphi(\hat{h}_1(x), \ldots, \hat{h}_d(x)) \), by Proposition 13.2.1 we have \( L_2 f(x) = (\mathcal{L}(d)\varphi)(\hat{h}_1(x), \ldots, \hat{h}_d(x)) \), where \( \mathcal{L}(d) \) is defined in (13.1.1). Recalling that \( \gamma \circ (\hat{h}_1, \ldots, \hat{h}_d)^{-1} = \gamma_d \), we get
\[
\int_X f^2 d\gamma = \int_{\mathbb{R}^d} \varphi^2 d\gamma_d, \quad \int_X (L_2 f)^2 d\gamma = \int_{\mathbb{R}^d} (\mathcal{L}(d)\varphi)^2 d\gamma_d,
\]
and, using (13.2.1),
\[
\|f\|_{W^{2,2}(X, \gamma)} = \|\varphi\|_{W^{2,2}(\mathbb{R}^d, \gamma_d)}.
\]
Therefore, estimates (13.1.7) imply that \( f \) satisfies (13.2.3), and the claim is proved.

The statement is now a standard consequence of the density of \( \Sigma \) in \( W^{2,2}(X, \gamma) \) and in \( D(L_2) \). Indeed, to prove that \( W^{2,2}(X, \gamma) \subset D(L_2) \), and that \( L_2 f = \text{div}_E \nabla_H f \) for every \( f \in W^{2,2}(\mathbb{R}^d, \gamma_d) \), it is sufficient to approximate any \( f \in W^{2,2}(X, \gamma) \) by a sequence \( (f_n) \) of elements of \( \Sigma \); then \( f_n \) converges to \( f \) and \( L_2 f_n = \text{div}_E \nabla_H f_n \) converges to \( \text{div}_E \nabla_H f \) in \( L^2(X, \gamma) \) by Theorem 10.2.7, as \( \nabla_H f_n \) converges to \( \nabla_H f \) in \( L^2(\gamma, H) \). Since \( L_2 \) is a closed operator, \( f \in D(L_2) \) and \( L_2 f = \text{div}_E \nabla_H f \). Similarly, to prove that \( D(L_2) \subset W^{2,2}(X, \gamma) \) we approximate any \( f \in D(L_2) \) by a sequence \( (f_n) \) of elements of \( \Sigma \) that converges to \( f \) in the graph norm; then \( (f_n) \) is a Cauchy sequence in \( W^{2,2}(X, \gamma) \) and therefore \( f \in W^{2,2}(X, \gamma) \).

Eventually, as in finite dimension, we have a characterisation of \( L_2 \) in terms of the bilinear form
\[
\mathcal{Q}(u, v) = \int_X [\nabla_H u, \nabla_H v]_H d\gamma, \quad u, v \in W^{1,2}(X, \gamma).
\]
Applying Theorem 13.1.5 with \( W = L^2(X, \gamma) \), \( V = W^{1,2}(X, \gamma) \) we obtain

Theorem 13.2.4. Let \( A \) be the operator associated with the bilinear form \( \mathcal{Q} \) above. Then \( D(A) = W^{2,2}(X, \gamma) \), and \( A = L_2 \).

The proof is identical to the proof of Theorem 13.1.6 and it is omitted.

Note that Theorem 13.2.4 implies that for every \( f \in D(L_2) = W^{2,2}(X, \gamma) \) and for every \( g \in W^{1,2}(X, \gamma) \) we have
\[
\int_X L_2 f g d\gamma = -\int_X [\nabla_H f, \nabla_H g]_H d\gamma
\]
that is the infinite dimensional version of (13.1.5). Proposition 11.4.3 implies that $L_2$ is a sectorial operator, therefore the Ornstein-Uhlenbeck semigroup is analytic in $L^2(X, \gamma)$.

We mention that, by general results about semigroups and interpolation theory (e.g. [D, Thm. 1.4.2]), $\{T_p(t) : p \geq 0\}$ is an analytic semigroup in $L^p(X, \gamma)$ for every $p \in (1, +\infty)$. However, this fact will not be used in these lectures.

A result similar to Theorem 13.2.3 holds also for $p \neq 2$. More precisely, for every $p \in (1, +\infty)$, $D(L_p) = W^{2,p}(X, \gamma)$, and the graph norm of $D(L_p)$ is equivalent to the $W^{2,p}$ norm. But the proof is not as simple. We refer to [M] and [B, Sect. 5.5] for the infinite dimensional case, and to [MPRS] for an alternative proof in the finite dimensional case.

### 13.3 Exercises

**Exercise 13.1.** Let $\varrho : \mathbb{R}^d \to [0, +\infty)$ be a mollifier, i.e. a smooth function with support in $B(0, 1)$ such that
\[
\int_{B(0, 1)} \varrho(x) dx = 1.
\]
For $\varepsilon > 0$ set
\[
\varrho_\varepsilon(x) = \varepsilon^{-d} \varrho\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d.
\]
Prove that if $p \in [1, +\infty)$ and $f \in L^p(\mathbb{R}^d, \gamma_d)$, then
\[
f_\varepsilon(x) := f \ast \varrho_\varepsilon(x) = \int_{\mathbb{R}^d} f(y) \varrho_\varepsilon(x - y) dy,
\]
is well defined, belongs to $L^p(\mathbb{R}^d, \gamma_d)$ and converges to $f$ in $L^p(\mathbb{R}^d, \gamma_d)$ as $\varepsilon \to 0^+$.

**Exercise 13.2.** Prove that for every $k \in \mathbb{N}$, $k \geq 3$, $C^k_b(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, \gamma_d)$ and that $T(t) \in \mathcal{L}(C^k_b(\mathbb{R}^d))$ for every $t > 0$.

**Exercise 13.3.** Let $\{h_j : j \in \mathbb{N}\}$ be any orthonormal basis of $H$ contained in $R_\gamma(X^*)$. Prove that the set $\Sigma$ of the cylindrical functions of the type $f(x) = \varphi(h_1(x), \ldots, h_d(x))$ with $\varphi \in C^2_b(\mathbb{R}^d)$, for some $d \in \mathbb{N}$, is dense in $L^p(X, \gamma)$ and in $W^{2,p}(X, \gamma)$ for every $p \in [1, +\infty)$.

**Exercise 13.4.**

(i) With the help of Proposition 10.1.2, show that if $f \in W^{1,p}(X, \gamma)$ with $p \in [1, +\infty)$ is such that $\nabla f = 0$ a.e., then $f$ is a.e. constant.

(ii) Use point (i) to show that for every $p \in [1, +\infty)$ the kernel of $L_p$ consists of the constant functions.

(HINT: First of all, prove that $T(t)f = f$ for all $f \in D(L_p)$ such that $L_pf = 0$ and then pass to the limit as $t \to +\infty$ in (12.1.3))
Bibliography


