Lecture 6

The classical Wiener space

In this Lecture we present the classical Wiener space, which is the archetype of the structure we are describing. Indeed, any triple \((X, \gamma, H)\) (where \(X\) is a separable Banach space, \(\gamma\) is a Gaussian measure and \(H\) is the Cameron-Martin space) is called an abstract Wiener space. In the classical Wiener space the Banach space is that of continuous paths, \(X = C([0,1])\), and all the objects involved can be described explicitly. The Gaussian measure is the Wiener measure \(\gamma^W\) defined in Lecture 5, the covariance operator is the integral operator with kernel \(\min\{x,y\}\) on \([0,1]^2\) and both the Cameron-Martin space \(H\) and \(X^*_\gamma\) are spaces of functions defined on \([0,1]\).

6.1 The classical Wiener space

We start by considering the measure space \((X, \mathcal{B}(X), \gamma^W)\) where \(X = C([0,1])\), \(\mathcal{B}(X)\) is the Borel \(\sigma\)-algebra on \(X\) and \(\gamma^W = \mathbb{P}^W \circ \overline{B}^{-1}\) is the measure defined in Proposition 5.2.10.

We give the following approximation result for measures in terms of Dirac measures. For every real measure \(\mu \in \mathcal{M}([0,1])\) and \(n \in \mathbb{N}\) we set

\[
\mu_n = \mu(\{1\})\delta_1 + \sum_{i=0}^{2^n-1} \mu\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]\right)\delta_{\frac{i+1}{2^n}}. \tag{6.1.1}
\]

**Lemma 6.1.1.** The following statements hold:

(i) for every \(\mu \in \mathcal{M}([0,1])\) the sequence \((\mu_n)\) converges weakly to \(\mu\);

(ii) for every \(\mu, \nu \in \mathcal{M}([0,1])\) the sequence \((\mu_n \otimes \nu)\) converges weakly to \(\mu \otimes \nu\);

(iii) for every \(\mu, \nu \in \mathcal{M}([0,1])\) the sequence \((\mu_n \otimes \nu_n)\) converges weakly to \(\mu \otimes \nu\).

**Proof.** (i) Let us fix \(f \in C([0,1])\). Then

\[
\int_{[0,1]} f(x)\mu_n(dx) - \int_{[0,1]} f(x)\mu(dx) = \sum_{i=0}^{2^n-1} \int_{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]} \left(f\left(\frac{i+1}{2^n}\right) - f(x)\right)\mu(dx).
\]

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By the uniform continuity of \( f \), for every \( \varepsilon > 0 \) there is \( n_0 > 0 \) such that for \( n > n_0 \) we have \(|f(i + 1/2^n) - f(x)| < \varepsilon \) for every \( x \in [i2^{-n}, (i + 1)2^{-n}] \) and for every \( i = 0, \ldots, 2^n - 1 \), whence for \( n > n_0 \)

\[
\left| \sum_{i=0}^{2^n-1} \int_{[i/2^n, i+1/2^n]} \left( f\left( \frac{i+1}{2^n} \right) - f(x) \right) \mu(dx) \right| < \varepsilon |\mu([0, 1])|
\]

and \( \mu_n(f) \rightarrow \mu(f) \).

(ii) Let now \( f \in C([0, 1]^2) \). Setting

\[
\varphi_n(y) = \int_{[0, 1]} f(x, y) \mu_n(dx), \quad \varphi(y) = \int_{[0, 1]} f(x, y) \mu(dx),
\]

we have

\[
\int_{[0, 1]^2} f(x, y)(\mu_n \otimes \nu)(d(x, y)) = \int_{[0, 1]} \varphi_n(y) \nu(dy).
\]

For any \( y \in [0, 1] \), \( \varphi_n(y) \) converges to \( \varphi(y) \) by statement (i). Since

\[
|\varphi_n(y)| \leq \|f\|_{\infty} |\mu|([0, 1]),
\]

by the Lebesgue Dominated Convergence Theorem we have

\[
\lim_{n \to +\infty} \int_{[0, 1]} \varphi_n(y) \nu(dy) = \int_{[0, 1]} \varphi(y) \nu(dy) = \int_{[0, 1]^2} f(x, y)(\mu \otimes \nu)(d(x, y)).
\]

(iii) Let now \( f \in C([0, 1]^2) \). Let \( \varphi_n \) and \( \varphi \) the functions in (ii), then

\[
\int_{[0, 1]^2} f(x, y)(\mu_n \otimes \nu)(d(x, y)) = \int_{[0, 1]} \varphi_n(x) \nu_n(dx)
\]

\[
= \int_{[0, 1]} \varphi(x) \nu_n(dx) + \int_{[0, 1]} (\varphi_n(x) - \varphi(x)) \nu_n(dx).
\]

We argue as in (i): by the uniform continuity of \( f \) for any \( \varepsilon > 0 \) there is \( n_0 > 0 \) such that for \( n > n_0 \)

\[
\sup_{x \in [0, 1]} |\varphi_n(x) - \varphi(x)| \leq \sum_{i=0}^{2^n-1} \int_{[i/2^n, i+1/2^n]} \left| f\left( \frac{i+1}{2^n} \right) - f(x, y) \right| |\mu|(dy) < \varepsilon |\mu|([0, 1]).
\]

Using the inequality \(|\nu_n|([0, 1]) \leq |\nu|([0, 1])\), see Exercise 6.1, we have

\[
\left| \int_{[0, 1]} (\varphi_n(x) - \varphi(x)) \nu_n(dx) \right| \leq \varepsilon |\mu|([0, 1]) \sup_{n \in \mathbb{N}} |\nu_n|([0, 1]) \leq \varepsilon |\nu|([0, 1]) |\mu|([0, 1])
\]

so that

\[
\lim_{n \to +\infty} \int_{[0, 1]^2} f(x, y)(\mu_n \otimes \nu)(d(x, y)) = \int_{[0, 1]^2} f(x, y)(\mu \otimes \nu)(d(x, y)).
\]
Statements (ii) and (iii) in Lemma 6.1.1 can be shown in a different way and for arbitrary sequences \((\mu_n)\) and \((\nu_n)\) weakly convergent to \(\mu\) and \(\nu\) respectively, see Exercise 6.2.

**Proposition 6.1.2.** The characteristic function of the Wiener measure \(\gamma^W\) is
\[
\widehat{\gamma^W}(\mu) = \exp\left\{-\frac{1}{2} \int_{[0,1]^2} \min\{x, y\}(\mu \otimes \mu)(dx, dy)\right\}, \quad \mu \in \mathcal{M}([0, 1]).
\]

So, \(\gamma^W\) is a Gaussian measure with mean zero and covariance operator
\[
B_{\gamma^W}(\mu, \nu) = \int_{[0,1]^2} \min\{t, s\}(\mu \otimes \nu)(d(t, s)), \quad \mu, \nu \in \mathcal{M}([0, 1]). \quad (6.1.2)
\]

**Proof.** We start by considering a linear combination of two Dirac measures
\[
\mu = \alpha \delta_s + \beta \delta_t
\]
with \(\alpha, \beta \in \mathbb{R}, s < t \in [0, 1].\) Then
\[
\widehat{\gamma^W}(\mu) = \int_X \exp\left\{i\alpha \delta_s(f) + i\beta \delta_t(f)\right\} \gamma^W(df) = \int_X \exp\left\{i\alpha f(s) + i\beta f(t)\right\} \gamma^W(df).
\]
Since \(\gamma^W = \mathbb{P}^W_A \circ \widehat{B}^{-1}\) with \(\mathbb{P}^W(A) = 1\) and \(\widehat{B}\) is a version of the Brownian motion which is continuous on \(A,\) noticing that \((\widehat{B}_t)_{t \in [0, 1]}\) and \((B_t)_{t \in [0, 1]}\) have the same image measure, we obtain
\[
\widehat{\gamma^W}(\mu) = \int_A \exp\left\{i\alpha \widehat{B}(\omega)(s) + i\beta \widehat{B}(\omega)(t)\right\} \mathbb{P}^W(d\omega) = \int_A \exp\left\{i\alpha \widehat{B}_s(\omega) + i\beta \widehat{B}_t(\omega)\right\} \mathbb{P}^W(d\omega)
\]
\[
= \int_{\mathbb{R}[0,1]} \exp\left\{i\alpha \widehat{B}_s(\omega) + i\beta \widehat{B}_t(\omega)\right\} \mathbb{P}^W(d\omega) = \int_{\mathbb{R}[0,1]} \exp\left\{i\alpha B_s(\omega) + i\beta B_t(\omega)\right\} \mathbb{P}^W(d\omega)
\]
\[
= \int_{\mathbb{R}[0,1]} \exp\left\{i(\alpha + \beta)B_s(\omega) + i\beta (B_t(\omega) - B_s(\omega))\right\} \mathbb{P}^W(d\omega).
\]
Since \(B_s\) and \(B_t - B_s\) are independent, we may write
\[
\widehat{\gamma^W}(\mu) = \int_{\mathbb{R}[0,1]} \exp\left\{i(\alpha + \beta)B_s(\omega) + i\beta (B_t(\omega) - B_s(\omega))\right\} \mathbb{P}^W(d\omega)
\]
\[
= \int_{\mathbb{R}[0,1]} \exp\left\{i(\alpha + \beta)B_s(\omega)\right\} \mathbb{P}^W(d\omega) \cdot \int_{\mathbb{R}[0,1]} \exp\left\{i\beta (B_t(\omega) - B_s(\omega))\right\} \mathbb{P}^W(d\omega)
\]
\[
= \int_{\mathbb{R}} \exp\left\{i(\alpha + \beta)x\right\} \mathcal{N}(0, s)(dx) \cdot \int_{\mathbb{R}} \exp\left\{i\beta y\right\} \mathcal{N}(0, t - s)(dy)
\]
\[
= \exp \left\{-\frac{1}{2}(\alpha + \beta)^2 s\right\} \exp \left\{-\frac{1}{2}\beta^2(t - s)\right\} \exp \left\{-\frac{1}{2}((\alpha^2 + 2\alpha\beta)s + \beta^2 t)\right\}.
\]
We now compute the integral
\[
\frac{1}{2} \int_{[0,1]^2} \min\{x, y\}(\mu \otimes \mu)(dx, dy) = \int_{[0,1]} \varphi(x)\mu(dx) = \alpha \varphi(s) + \beta \varphi(t),
\]
where \( \varphi(x) = \int_{[0,1]} \min\{x, y\} \mu(dy) \). We have

\[
\varphi(s) = \int_{[0,1]} \min\{s, y\} \mu(dy) = \int_{[0,s]} y \mu(dy) + s \int_{(s,1]} \mu(dy) = \alpha s + \beta s
\]

\[
\varphi(t) = \int_{[0,1]} \min\{t, y\} \mu(dy) = \int_{[0,t]} y \mu(dy) + t \int_{(t,1]} \mu(dy) = \alpha s + \beta t,
\]

whence

\[
\int_{[0,1]^2} \min\{x, y\} (\mu \otimes \mu)(d(x, y)) = (\alpha^2 + 2\alpha \beta) s + \beta^2 t.
\]

So, the assertion of the theorem holds if \( \mu \) is a linear combination of two Dirac measures. The same assertion holds true if \( \mu \) is a linear combination of a finite number of Dirac measures. In the general case we conclude by using Lemma 6.1.1. Indeed, if \( \mu_n \) is the approximation of \( \mu \) defined in (6.1.1), then for any \( f \in X \),

\[
\lim_{n \to +\infty} \mu_n(f) = \lim_{n \to +\infty} \int_{[0,1]} f(x) \mu_n(dx) = \int_{[0,1]} f(x) \mu(dx) = \mu(f).
\]

Hence \( \exp\{i\mu_n(f)\} \) converges to \( \exp\{i\mu(f)\} \) for any \( f \in X \), so by the Lebesgue Dominated Convergence Theorem

\[
\gamma^W(\mu) = \int_X \exp\{i\mu(f)\} \gamma^W(df) = \lim_{n \to +\infty} \int_X \exp\{i\mu_n(f)\} \gamma^W(df)
\]

\[
= \lim_{n \to +\infty} \exp \left\{ -\frac{1}{2} \int_{[0,1]^2} \min\{x, y\} (\mu_n \otimes \mu_n)(d(x, y)) \right\}
\]

\[
= \exp \left\{ -\frac{1}{2} \int_{[0,1]^2} \min\{x, y\} (\mu \otimes \mu)(d(x, y)) \right\}.
\]

Then we conclude applying Theorem 2.2.4. \( \square \)

We notice that the space

\[
C_0([0,1]) := \{ f \in C([0,1]) : f(0) = 0 \} = \delta_0^{-1}({0})
\]

is a closed subspace of \( C([0,1]) \). Since

\[
\gamma^W(\delta_0) = \exp \left\{ -\frac{1}{2} \int_{[0,1]^2} \min\{x, y\} (\delta_0 \otimes \delta_0)(d(x, y)) \right\} = 1,
\]

\( \gamma^W \circ \delta_0^{-1} = \mathcal{N}(0, 0) = \delta_0 \), and so

\[
\gamma^W(C_0([0,1])) = (\gamma^W \circ \delta_0^{-1})(\{0\}) = 1.
\]

Then \( \gamma^W \) is degenerate and it is concentrated on \( C_0([0,1]) \).
6.2 The Cameron–Martin space

In order to characterise the Cameron–Martin space of \((C([0,1]),\gamma^W)\), we use the embedding \(\iota : C([0,1]) \to L^2(0,1), \iota(f) = f\), which is a continuous injection since

\[ \|\iota(f)\|_{L^2(0,1)} \leq \|f\|_{\infty}. \]

If we consider the image measure \(\tilde{\gamma}^W := \gamma^W \circ \iota^{-1}\) on \(L^2(0,1)\), the Cameron–Martin spaces on \(C([0,1])\) and on \(L^2(0,1)\) are the same in the sense of Proposition 3.1.10. The fact that \(L^2(0,1)\) is a Hilbert space allows us to use the results of Section 4.2.

By using the identification \((L^2(0,1))^* = L^2(0,1)\), the characteristic function of the Gaussian measure \(\tilde{\gamma}^W\) is

\[ \hat{\tilde{\gamma}}^W(g) = \int_{L^2(0,1)} \exp\{i(f,g)_{L^2(0,1)}\} \tilde{\gamma}^W(df) = \int_{C([0,1])} \exp\{i(\iota(f),g)_{L^2(0,1)}\} \gamma^W(df). \]

Let us compute \(\langle \iota(f),g\rangle_{L^2(0,1)}\). If we denote by \(\iota^* : L^2(0,1) \to \mathcal{M}([0,1])\) the adjoint of \(\iota\), then

\[ \langle \iota(f),g\rangle_{L^2(0,1)} = \iota^*(g)(f). \] (6.2.1)

Since \(\iota(f)(x) = f(x)\), (6.2.1) yields

\[ \int_0^1 f(x)g(x)dx = \int_{[0,1]} f(x)\iota^*(g)(dx), \quad \forall f \in C([0,1]). \]

Hence \(\iota^*(g) = g\lambda_1\), where \(\lambda_1\) is the Lebesgue measure on \([0,1]\). Therefore, according to Proposition 6.1.2,

\[ \hat{\tilde{\gamma}}^W(g) = \hat{\gamma}^W(\iota^* g) = \exp\left\{ -\frac{1}{2} \int_{[0,1]^2} \min\{x,y\} g(x)g(y) d(x,y) \right\} \]

so that \(\tilde{\gamma}^W\) is a Gaussian measure with covariance

\[ B_{\tilde{\gamma}^W}(f,g) = \int_{[0,1]^2} \min\{x,y\} f(x)g(y) d(x,y) = \int_0^1 Qf(y)g(y)dy, \]

where

\[ Qf(y) = \int_0^1 \min\{x,y\} f(x)dx \]

is the covariance operator \(Q : L^2(0,1) \to L^2(0,1)\) introduced in Section 4.2.

**Theorem 6.2.1.** The Cameron–Martin space \(H\) of \(\tilde{\gamma}^W\) on \((L^2(0,1),\mathcal{B}(L^2(0,1)))\) is

\[ H_0^1([0,1]) := \{ f \in L^2(0,1) : f' \in L^2(0,1) \text{ and } f(0) = 0 \}. \]
Proof. As the Cameron-Martin space is the range of $Q^{1/2}$, see Theorem 4.2.6, we find the eigenvalues and eigenvectors of $Q$, i.e., we look for all $\lambda \in \mathbb{R}$ and $f \in L^2(0,1)$ such that $Qf = \lambda f$. Equality $Qf = \lambda f$ is equivalent to

$$\lambda f(x) = \int_0^1 \min\{x, y\} f(y) dy = \int_0^x yf(y) dy + x \int_x^1 f(y) dy \quad (6.2.2)$$

for a.e. $x \in [0,1]$. If (6.2.2) holds, $f$ is weakly differentiable with

$$\lambda f'(x) = \int_x^1 f(y) dy.$$

Moreover, the continuous version of $f$ vanishes at 0. Hence, in its turn, $f'$ is weakly differentiable and

$$\lambda f''(x) = -f(x) \quad \text{a.e.}$$

The continuous version of $f'$ vanishes at 1. We have proved that if $f$ is an eigenvector of $Q$ with eigenvalue $\lambda$, then $f$ is the solution of the following problem on $(0,1)$:

$$\begin{cases} 
\lambda f'' + f = 0, \\
f(0) = 0, \\
f'(1) = 0.
\end{cases} \quad (6.2.3)$$

On the other hand, if $f$ is the solution of problem (6.2.3), integrating between $x$ and 1

$$\lambda f'(x) = \int_x^1 f(y) dy,$$

whence, integrating again between 0 and $x$

$$\lambda f(x) = \int_0^x \int_t^1 f(y) dy dt = \int_0^x \mathbb{I}_{(0,x]}(t) \int_0^1 \mathbb{I}_{(t,1]}(y)f(y) dy dt$$

$$= \int_0^1 f(y) \int_0^x \mathbb{I}_{(0,x]}(t) \mathbb{I}_{(t,1]}(y) dt dy = \int_0^1 f(y) \int_0^1 \mathbb{I}_{(t,1]}(x) \mathbb{I}_{(t,1]}(y) dt dy$$

$$= \int_0^1 \min\{x, y\} f(y) dy = Qf(x).$$

We leave as an exercise, see Exercise 6.3, the fact if $\lambda$ is an eigenvalue, then there exists $k \in \mathbb{N}$ such that $\lambda = \lambda_k$, where

$$\lambda_k = \frac{1}{\pi^2 \left( k + \frac{1}{2} \right)^2}, \quad k \in \mathbb{N} \quad (6.2.4)$$

and $Qe_k = \lambda_k e_k$, $\|e_k\|_{L^2(0,1)} = 1$ if and only if

$$e_k(x) = \sqrt{2} \sin \left( \frac{x}{\sqrt{\lambda_k}} \right) = \sqrt{2} \sin \left( \frac{2k + 1}{2} \pi x \right). \quad (6.2.5)$$
Let us now take \( f \in L^2(0, 1) \) and write
\[
f = \sum_{k=1}^{\infty} f_k e_k, \quad f_k = \langle f, e_k \rangle_{L^2(0, 1)} \tag{6.2.6}
\]
with \( e_k \) be given by (6.2.5). Applying (4.2.9), we see that \( f \in H \) if and only if
\[
\sum_{k=1}^{\infty} f_k^2 \lambda_k^{-1} = \pi^2 \sum_{k=1}^{\infty} f_k^2 \left( \frac{2k+1}{2} \right)^2 < +\infty.
\]
Such condition allows us to define the function
\[
g(x) = \sqrt{2} \sum_{k=1}^{\infty} \frac{f_k}{\sqrt{\lambda_k}} \cos \left( \frac{x}{\sqrt{\lambda_k}} \right) = \pi \sqrt{2} \sum_{k=1}^{\infty} f_k \frac{2k+1}{2} \cos \left( x \pi \left( \frac{2k+1}{2} \right) \right)
\]
and to obtain that \( g \in L^2(0, 1) \) is the weak derivative of \( f \). Indeed, for any \( \varphi \in C_c^\infty((0, 1)) \),
\[
\int_0^1 f(x) \varphi'(x) dx = \sum_{k=1}^{\infty} f_k \sqrt{2} \int_0^1 \sin \left( \frac{s}{\sqrt{\lambda_k}} \right) \varphi'(x) dx
\]
\[
= -\sum_{k=1}^{\infty} f_k \sqrt{2} \int_0^1 \cos \left( \frac{x}{\sqrt{\lambda_k}} \right) \varphi(x) dx = -\int_0^1 g(x) \varphi(x) dx.
\]
In conclusion \( f \in H^1(0, 1) \) and, by (6.2.6), its continuous version vanishes at 0, whence \( f \in H_0^1([0, 1]) \).

Eventually, from the equality
\[
|f|_H = \|Q^{-1/2} f\|_X,
\]
we immediatly get \( |f|_H = \|f'\|_{L^2(0, 1)} \).

**Remark 6.2.2.** We have used the notation \( H_0^1([0, 1]) \) to characterise the Cameron–Martin space; we point out that this space is not the closure of \( C_c^\infty(0, 1) \) in \( H^1(0, 1) \).

### 6.3 The reproducing kernel

In this section we characterise the reproducing kernel, both for \( X = C([0, 1]) \) and for \( X = L^2(0, 1) \). To do this, we need to introduce an important tool coming from probability, the stochastic integral (the Itô integral).

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \((W_t)_{t \in [0, 1]}\) be a Brownian motion, i.e., a stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying the conditions of Definition 5.2.4.

If \( f \) is a simple function, i.e.,
\[
f(t) = \sum_{i=0}^{n-1} c_i \mathbb{I}_{[t_i, t_{i+1})}(t)
\]
with \( c_i \in \mathbb{R}, 0 = t_0 < \ldots < t_n = 1 \), we define the random variable on \( \Omega \)
\[
\int_0^1 f(t) dW_t(\omega) := \sum_{i=0}^{n-1} c_i (W_{t_{i+1}}(\omega) - W_{t_i}(\omega)).
\] (6.3.1)

We claim that
\[
\mathbb{E} \left[ \left( \int_0^1 f(t) dW_t \right)^2 \right] = \int_0^1 f(x)^2 dx.
\] (6.3.2)

Indeed, we have
\[
\mathbb{E} \left[ \left( \int_0^1 f(t) dW_t \right)^2 \right] = \sum_{i,j=0}^{n-1} c_i c_j \mathbb{E}[(W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})]
\]
\[
= \sum_{i=0}^{n-1} c_i^2 \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] + 2 \sum_{i=1}^{n-1} \sum_{j<i} c_i c_j \mathbb{E}[(W_{t_{j+1}} - W_{t_j})(W_{t_{i+1}} - W_{t_i})]
\]
\[
= \sum_{i=0}^{n-1} c_i^2 \mathbb{E}[(W_{t_{i+1}} - t_i - t_i)^2] + 2 \sum_{i=1}^{n-1} \sum_{j<i} c_i c_j \mathbb{E}[W_{t_i+1} - W_{t_i}] \cdot \mathbb{E}[W_{t_{j+1}} - W_{t_j}]
\]
\[
= \sum_{i=0}^{n-1} c_i^2 (t_{i+1} - t_i) = \int_0^1 |f(x)|^2 dx,
\]
where we have used the fact that \( W_{t_{i+1}} - W_{t_i} \) is independent of \( W_{t_{j+1}} - W_{t_j} \) if \( j < i \) and the fact that \( W_{t_{i+1}} - W_{t_i} \) has the same image measure as \( W_{t_{i+1} - t_i} \), given by \( \mathcal{N}(0, t_{i+1} - t_i) \).

For the next Theorem, we refer to [B, Section 2.11].

**Theorem 6.3.1 (Itô Integral).** There exists a unique map \( I_\Omega : L^2(0, 1) \to L^2(\Omega, \mathbb{P}) \) such that
\[
\mathbb{E}[|I_\Omega(f)|^2] = \int_0^1 |f(x)|^2 dx, \quad \forall f \in L^2(0, 1)
\] (6.3.3)

and such that
\[
I_\Omega(f) = \int_0^1 f(t) dW_t
\]
if \( f \) is a simple function. Such a map is called the Itô integral of \( f \), identity (6.3.3) is called Itô isometry and the Itô integral is denoted by
\[
I_\Omega(f) = \int_0^1 f(t) dW_t, \quad f \in L^2(0, 1).
\]

**Proof.** Let \( \mathcal{S}([0, 1]) \) be the linear subspace of \( L^2(0, 1) \) consisting of the simple functions. The map \( I_\Omega : \mathcal{S}([0, 1]) \to L^2(\Omega, \mathbb{P}) \),
\[
I_\Omega(f)(\omega) := \int_0^1 f(t) dW_t(\omega)
\]
is a linear operator, defined on a dense subset of \( L^2(0, 1) \). Since it is continuous in the \( L^2(0, 1) \) topology by (6.3.2), it has a unique continuous extension to \( L^2(0, 1) \). \( \square \)
In order to apply Theorem 6.3.1, we need to define Brownian motions on the probability spaces \((C([0, 1]), \mathcal{B}(C([0, 1])), \gamma^W)\) and \((L^2(0, 1), \mathcal{B}(L^2(0, 1)), \tilde{\gamma}^W)\). We leave as an exercise (see Exercise 6.4) the verification that the evaluation maps \(W_t : C([0, 1]) \to \mathbb{R}\) and \(\tilde{W}_t : L^2(0, 1) \to \mathbb{R}\) given for continuous functions by

\[
W_t(f) := f(t), \quad \tilde{W}_t(f) := f(t)
\]

are indeed standard Brownian motions on \(C([0, 1])\) and on \(L^2(0, 1)\) respectively.

We pass now to the characterisation of the reproducing kernels \(X^*_\gamma^W\) and \(X^*_\tilde{\gamma}^W\).

**Proposition 6.3.2.** Let us consider the Gaussian measures \(\gamma^W\) and \(\tilde{\gamma}^W\) on \(C([0, 1])\) and on \(L^2(0, 1)\), respectively. Then

\[
X^*_\gamma^W = I_{C([0, 1])}(L^2(0, 1))
\]

and

\[
X^*_\tilde{\gamma}^W = I_{L^2(0, 1)}(L^2(0, 1)).
\]

**Proof.** Let us consider the simple function

\[
g(x) = \alpha \mathbb{1}_{[0, s]}(x) + \beta \mathbb{1}_{[0, t]} = (\alpha + \beta) \mathbb{1}_{[0, s]}(x) + \beta \mathbb{1}_{[s, t]}(x),
\]

\(s < t \in [0, 1]\). Let \((W_t)\) be the Brownian motion defined in (6.3.4). Then by (6.3.1)

\[
I_{C([0, 1])}(g) = \int_0^1 g(t) dW_t = (\alpha + \beta)W_s + \beta(W_t - W_s) = \alpha W_s + \beta W_t.
\]

Therefore,

\[
I_{C([0, 1])}(g)(f) = \alpha f(s) + \beta f(t)
\]

for \(\gamma^W\)-a.e. \(f \in C([0, 1])\). On the other hand, setting

\[
\mu = \alpha \delta_s + \beta \delta_t,
\]

since \(j(\mu)(f) = \alpha f(s) + \beta f(t)\) for any \(f \in C([0, 1])\), we obtain \(I_{C([0, 1])}(g) = j(\mu)\) \(\gamma^W\)-a.e..

Moreover, by (6.3.3)

\[
\|g\|_{L^2(0, 1)}^2 = \mathbb{E} \left[ (I_{C([0, 1])}(g))^2 \right] = \|I_{C([0, 1])}(g)\|_{L^2(C([0, 1]), \gamma^W)}^2
\]

\[
= \|j(\mu)\|_{L^2(C([0, 1]), \gamma^W)}^2.
\]

For any simple function \(g \in \mathcal{S}([0, 1])\)

\[
g(x) = \sum_{i=1}^n c_i \mathbb{1}_{[0, t_i]}(x),
\]

a similar computation yields \(\|g\|_{L^2(0, 1)} = \|j(\mu)\|_{L^2(C([0, 1]), \gamma^W)}\) with

\[
\mu = \sum_{i=1}^n c_i \delta_{t_i},
\]
and \( I_{C([0,1])}(g) = j(\gamma) \gamma^W \)-a.e. Approaching any \( g \in L^2(0,1) \) by a sequence of simple functions \( g_n \), \( I_{C([0,1])}(g_n) \) converges to \( I_{C([0,1])}(g) \) in \( L^2(C([0,1]), \gamma^W) \) by the Itô isometry (6.3.3). Since \( I_{C([0,1])}(g_n) \in j(M([0,1])) \) for every \( n \in \mathbb{N} \), we have \( I_{C([0,1])}(g) \in X^*_W \).

This proves that \( I_{C([0,1])}(L^2(0,1)) \subset X^*_W \).

For the reverse inclusion, we use Lemma 6.1.1. Let us take \( \mu \in M([0,1]) \) and let \( (\mu_n) \) be the approximating sequence defined by (6.1.1). Then \( \gamma^W \)-a.e. \( j(\mu_n) = I_{C([0,1])}(g_n) \), where

\[
g_n(x) = \mu(\{1\}) + \sum_{i=0}^{2^n-1} \mu \left( \left[ \frac{i}{2^n}, \frac{i+1}{2^n} \right) \right) \mathbb{I}_{[0,1]}(x)
\]

\[
= \mu(\{1\}) + \sum_{i=0}^{2^n-1} \mu \left( \left[ \frac{i}{2^n}, \frac{i+1}{2^n} \right) \right) \sum_{j=0}^{i} \mathbb{I}_{[j2^n, j+12^n)}(x)
\]

\[
= \sum_{j=0}^{2^n-1} \mu(\{1\}) \mathbb{I}_{[j2^n, j+12^n)}(x) + \sum_{j=0}^{2^n-1} \sum_{i=j}^{2^n-1} \mu \left( \left[ \frac{i}{2^n}, \frac{i+1}{2^n} \right) \right) \mathbb{I}_{[j2^n, j+12^n)}(x)
\]

\[
= \sum_{j=0}^{2^n-1} c_{n,j} \mathbb{I}_{[j2^n, j+12^n)}(x),
\]

where

\[
c_{n,j} = \mu(\{1\}) + \sum_{i=j}^{2^n-1} \mu \left( \left[ \frac{i}{2^n}, \frac{i+1}{2^n} \right) \right) = \mu \left( \left[ \frac{j}{2^n}, 1 \right) \right).
\]

The functions \( j(\mu_n) \) converge to \( j(\mu) \) in \( L^2(C([0,1]), \gamma^W) \) since

\[
\|j(\mu_n) - j(\mu)\|^2_{L^2(C([0,1]), \gamma^W)} = \|j(\mu_n - \mu)\|_{L^2(C([0,1]), \gamma^W)}^2 = B_{\gamma^W}(\mu_n - \mu, \mu_n - \mu)
\]

\[
= \int_{[0,1]^2} \min\{x, y\} ((\mu_n - \mu) \otimes (\mu_n - \mu)) (d(x, y))
\]

\[
= \int_{[0,1]^2} \min\{x, y\} (\mu_n \otimes \mu_n) (d(x, y)) - 2 \int_{[0,1]^2} \min\{x, y\} (\mu_n \otimes \mu) (d(x, y))
\]

\[
+ \int_{[0,1]^2} \min\{x, y\} (\mu \otimes \mu) (d(x, y)).
\]

As \( \mu_n \) converges to \( \mu \) weakly, \( \mu_n \otimes \mu \) and \( \mu_n \otimes \mu_n \) converge to \( \mu \otimes \mu \) weakly, we obtain

\[
\lim_{n \to +\infty} \|j(\mu_n) - j(\mu)\|_{L^2(C([0,1]), \gamma^W)}^2 = 0.
\]

Hence \( j(\mu) \) is the limit in \( L^2(C([0,1]), \gamma^W) \) of \( j(\mu_n) = I_{C([0,1])}(g_n) \), whence \( j(M([0,1])) \subset I_{C([0,1])}(L^2(0,1)) \). Hence \( X^*_W \subset I_{C([0,1])}(L^2(0,1)) \), and this concludes the proof of the equality \( X^*_W = I_{C([0,1])}(L^2(0,1)) \).

The proof of equality \( X^*_W = I_{L^2(0,1)}(L^2(0,1)) \) is similar. \( \square \)
Remark 6.3.3. By Proposition 3.1.2, a function \( f \in C([0,1]) \) belongs to the Cameron–Martin space \( H \) if and only if it belongs to the range of \( R_{\gamma W} \), namely if and only if \( f = R_{\gamma W} (IC([0,1])(g)) \) for some \( g \in L^2(0,1) \). In this case, by (3.1.4) we have

\[
|f|_H = \|IC([0,1])(g)\|_{L^2(C([0,1]),\gamma W)},
\]

and by the Itô isometry (6.3.3)

\[
|f|_H = \|g\|_{L^2(0,1)}.
\]

The same argument holds true for \( f \in L^2(0,1) \).

We close this lecture by characterising the spaces

\[
R_{\gamma W}(j(M([0,1])))
\]

and

\[
R_{\tilde{\gamma} W}(j(L^2(0,1))) = Q(L^2(0,1)).
\]

The latter is easier to describe. We have indeed the following result.

Proposition 6.3.4. Let \( \tilde{\gamma} W \) be the Wiener measure on \( L^2(0,1) \). Then

\[
Q(L^2(0,1)) = \{ u \in H^1_0([0,1]) \cap H^2(0,1) : u'(1) = 0 \}
\]

and for any \( f \in L^2(0,1) \), \( u = Qf \) is the solution of the problem on \( (0,1) \)

\[
\begin{cases}
  u'' = -f \\
  u(0) = 0, \\
  u'(1) = 0.
\end{cases}
\]

(6.3.6)

Proof. If \( u = Qf \), then

\[
u(x) = Qf(x) = \int_0^1 \min\{x, y\} f(y) dy.
\]

Then \( u \) is weakly differentiable and

\[
u'(x) = \int_x^1 f(y) dy.
\]

Hence \( u' \) admits a continuous version such that \( u'(1) = 0 \); \( u' \) is also a.e. differentiable with

\[
u''(x) = -f(x).
\]

On the other hand, arguing as in the proof of Proposition 6.2.1, if \( u \) is a solution of (6.3.6), integrating twice we obtain

\[
u(x) = \int_0^1 \min\{x, y\} f(y) dy = Qf(x),
\]

and this completes the proof.  \( \square \)
To prove a similar result in the case of $\gamma^W$ on $C([0, 1])$, we need the following lemma.

**Lemma 6.3.5.** Let $v \in L^2(0, 1)$ be such that $v' = \mu \in \mathcal{M}([0, 1])$ in the sense of distributions, i.e.

$$\int_0^1 v(x)\varphi'(x)\,dx = -\int_{[0,1]} \varphi(x)\mu(dx), \quad \forall \varphi \in C_c^1(0, 1).$$

Then there exists $c \in \mathbb{R}$ such that for a.e. $x \in (0, 1)$

$$v(x) = \mu((0, x]) + c. \quad (6.3.7)$$

**Proof.** Let us set $w(x) = \mu((0, x])$. We claim that $w' = \mu$ is the sense of distributions. Indeed, for any $\varphi \in C_c^1(0, 1)$, by the Fubini Theorem 1.1.15

$$\int_0^1 w(x)\varphi'(x)\,dx = \int_0^1 \mu((0, x])\varphi'(x)\,dx = \int_0^1 \left(\int_{(0,1)} \mathbb{1}_{(0,x]}(y)\mu(dy)\right)\varphi'(x)\,dx$$

$$= \int_{(0,1)} \left(\int_0^1 \mathbb{1}_{(0,x]}(y)\varphi'(x)\,dx\right)\mu(dy) = -\int_{(0,1)} \varphi(y)\mu(dy).$$

As a consequence, the weak derivative of $v - w$ is zero and the conclusion holds. \square

**Definition 6.3.6.** Let $v \in L^2(0, 1)$ be such that $v' \in \mathcal{M}([0, 1])$ in the sense of distributions. Then, writing $v(x) = \mu((0, x]) + c$ for a.e $x \in (0, 1)$ as in (6.3.7), we set

$$v(1^-) := c + \mu((0, 1)).$$

We close this lecture with the following proposition.

**Proposition 6.3.7.** Let $\gamma^W$ be the Wiener measure on $C([0, 1])$. Then

$$R_{\gamma^W}(j(\mathcal{M}([0, 1]))) = \{ u \in H^1_0([0, 1]) : \exists \mu \in \mathcal{M}([0, 1]) \text{ s.t. } u'' = -\mu \text{ on } (0, 1) \text{ in the sense of distributions, } u'(1^-) = \mu(\{1\}) \}$$

and $u = R_{\gamma^W}(j(\mu))$ if and only if $u$ is the solution of the following problem on $(0, 1)$

$$\begin{cases} u'' = -\mu, \\
u(0) = 0, \\
(1^-) = \mu(\{1\}). \quad (6.3.8) \end{cases}$$

**Proof.** Let $u = R_{\gamma^W}(j(\mu))$. Then for any $\nu \in \mathcal{M}([0, 1])$ we have

$$\nu(u) = \nu(R_{\gamma^W}(j(\mu))) = \int_{C([0, 1])} j(\mu)(f)j(\nu)(f)\gamma^W(df)$$

$$= \int_{C([0, 1])} \int_{[0,1]} f(x)\mu(dx) \int_{[0,1]} f(y)\nu(dy)\gamma^W(df)$$

$$= \int_{[0,1]^2} \left(\int_{C([0, 1])} W_x(f)W_y(f)\gamma^W(df)\right) (\mu \otimes \nu)(d(x, y))$$

$$= \int_{[0,1]^2} \min\{x, y\} (\mu \otimes \nu)(d(x, y)),$$
where we have used the fact that \( W_t \) is a standard Brownian motion. Indeed, since if \( x < y \), \( W_x \) and \( W_y - W_x \) are independent,

\[
\int_{C([0,1])} W_x(f) W_y(f) \gamma^W(df) = \int_{C([0,1])} W_x(f) (W_y(f) - W_x(f)) \gamma^W(df) + \\
+ \int_{C([0,1])} W_x(f)^2 \gamma^W(df)
\]

\[
= \int_{C([0,1])} W_x(f) \gamma^W(df) \cdot \int_{C([0,1])} (W_y(f) - W_x(f)) \gamma^W(df) + \\
+ \int_{C([0,1])} W_x(f)^2 \gamma^W(df) = x.
\]

Then

\[
u(y) = R_{y,w}(\mu(x)) = \int_{[0,1]} \min\{x,y\} \mu(dx).
\] (6.3.9)

We claim that for a.e. \( y \in (0,1) \)

\[
u'(y) = \mu((y,1]) = -\mu((0,y]) + \mu((0,1])
\] (6.3.10)

and that \( \nu' \in L^2(0,1) \). Indeed, for any \( \varphi \in C_c^1(0,1) \),

\[
\int_0^1 \nu(x) \varphi'(x) dx = \int_{[0,1]} \int_0^1 \min\{x,y\} \varphi'(x) dx \mu(dy) \\
= - \int_{[0,1]} \int_0^y \varphi(x) dx \mu(dy) = - \int_{[0,1]} \varphi(x) \int_{[0,1]} \mathbb{1}_{(0,y]}(x) \mu(dy) dx \\
= - \int_{[0,1]} \varphi(x) \int_{[0,1]} \mathbb{1}_{(x,1]}(y) \mu(dy) dx = - \int_{[0,1]} \varphi(x) \mu((x,1]) dx.
\]

It is readily verified that \( \nu'(x) = \mu((x,1]) \) belongs to \( L^2(0,1) \); in addition \( \nu'' = -\mu \) on \( (0,1) \) in the sense of distributions. Indeed for any \( \varphi \in C_c^1(0,1) \)

\[
\int_0^1 \nu'(x) \varphi'(x) dx = \int_{[0,1]} \int_0^1 \mathbb{1}_{(0,y]}(x) \varphi'(x) dx \mu(dy) \\
= \int_{[0,1]} \varphi(y) \mu(dy) = \int_{(0,1)} \varphi(y) \mu(dy),
\]

where the last equality follows from the fact that \( \varphi(0) = \varphi(1) = 0 \). Comparing (6.3.10) with (6.3.7) in Lemma 6.3.5, we obtain

\[
\lim_{y \to 1^-} \nu'(y) = \mu(\{1\}).
\]

On the other hand, if \( u \) is a solution of (6.3.8), then since \( u'' = -\mu \), by (6.3.7) in Lemma 6.3.5 we have for a.e. \( x \in (0,1) \)

\[
u'(x) = -\mu((0,x]) + c = \mu((x,1]) - \mu((0,1]) + c.
\]
Then $u'(1^-) = \mu(\{1\})$ if and only if $c = \mu(\{1\}) + \mu((0, 1))$, and in this case $u'(x) = \mu((x, 1])$ for a.e. $x \in (0, 1)$. Integrating $u'$ between 0 and $x$, we get

$$u(x) = \int_0^x \mu((t, 1])dt = \int_0^1 \mathbb{1}_{(0,x)}(t)\mu((t, 1])dt = \int_0^1 \int_{[0,1]} \mathbb{1}_{(0,x)}(t)\mathbb{1}_{(t,1]}(y)\mu(dy) dt$$

$$= \int_{[0,1]} \int_{[0,1]} \mathbb{1}_{(t,1]}(x)\mathbb{1}_{(t,1]}(y)\mu(dy) dt \mu(dx) = \int_{[0,1]} \min\{x,y\}\mu(dy),$$

and then $u = R_{\gamma}w(j(\mu))$. This completes the proof.

6.4 Exercises

**Exercise 6.1.** Let $\mu \in \mathcal{M}([0,1])$ and let $\mu_n$ be the sequence defined by (6.1.1); prove that $|\mu_n|([0,1]) \leq |\mu|([0,1])$.

**Exercise 6.2.** Show that if $(\mu_n)$ is a sequence of real measures on $(X, \mathcal{F}_1)$ weakly convergent to $\mu$ and $(\nu_n)$ is a sequence of measures on $(Y, \mathcal{F}_2)$ weakly convergent to $\nu$ then the sequence $(\mu_n \otimes \nu_n)$ converges weakly to $\mu \otimes \nu$, cf. statements (ii) and (iii) in Lemma 6.1.1.

**Exercise 6.3.** Verify that the eigenvalues of $Q$ in Theorem 6.2.1 are given by (6.2.4) and that the eigenfunctions with unit norm are given by (6.2.5).

**Exercise 6.4.** Prove that the stochastic processes defined in (6.3.4) are standard Brownian motions, i.e. they satisfy Definition 5.2.4.

**Exercise 6.5.** Prove that the function $g_n$ in formula (6.3.5) is given by

$$g_n(x) = \mu\left(\left[\frac{[2^n x]}{2^n}, 1\right]\right),$$

where $[2^n x]$ is the integer part of $2^n x$. Prove that if $\mu \in \mathcal{M}([0,1])$, $j(\mu) = I_C([0,1])(g)$ where

$$g(x) = \mu([x,1]), \quad \text{for a.e. } x \in [0,1].$$

**Exercise 6.6.** Prove that for any $\mu \in \mathcal{M}([0,1])$, the function

$$u(y) = \int_{[0,1]} \min\{x,y\}\mu(dx), \quad y \in [0,1],$$

is continuous.

Bibliography